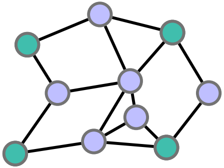
Lower bounds for multivariate independence polynomials *and* their generalizations



Jaehyeon Seo (Yonsei University)

Joint with Joonkyung Lee;
Ewan Davies, Juspreet Singh Sandhu, Brian Tan

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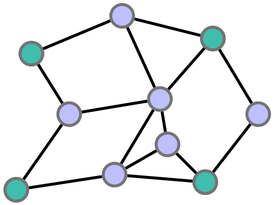
Basics

- Graph $G = (V, E)$
- Independent set



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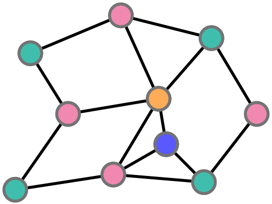
- Proper vertex coloring



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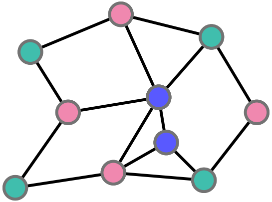
- Graph $G = (V, E)$
- Independent set
- Proper vertex coloring
- Semiproper vertex coloring (“looped” color )



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$\mathcal{G}$	Maximizer	Minimizer
d-regular bipartite	$K_{d,d}$ (Kahn '01)	infinite d-regular tree (Csikvári '16+)
d-regular	$K_{d,d}$ (Zhao '10)	K_{d+1} (Cutler–Radcliffe '14)
general graphs	bicliques (SSSZ '19)	cliques (SSSZ '19)

Number of independent sets: irregular graphs

Theorem (Sah–Sawhney–Stoner–Zhao '19)

Among the graphs with a **fixed degree distribution**, the disjoint union of cliques minimizes $i(G)^{1/v(G)}$.

$$\prod_{v \in V(G)} i(K_{d_v+1})^{1/(d_v+1)} \leq i(G)$$

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Theorem (Sah–Sawhney–Stoner–Zhao '19)

Among the graphs with a **fixed degree distribution**, the disjoint union of cliques minimizes $i(G)^{1/v(G)}$.

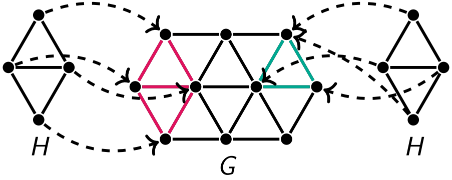
Among the graphs with a **fixed degree-degree distribution**, the disjoint union of bicliques maximizes $i(G)^{1/e(G)}$.

$$\prod_{v \in V(G)} i(K_{d_v+1})^{1/(d_v+1)} \leq i(G) \leq \prod_{uv \in E(G)} i(K_{d_u,d_v})^{1/(d_u d_v)}$$

Recall: Graph homomorphisms

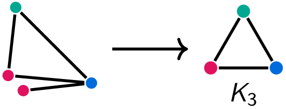
Graph homomorphism := vertex map which preserves edges

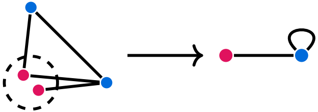
Hom(H, G) := {the homomorphisms $H \to G$}



Graph homomorphisms represent several canonical objects:

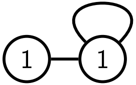
- Hom(H, K_q) = {the proper q-vertex-colorings of H}
- Hom(H, ) = {the independent sets of H}



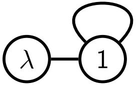


Independence polynomials

$\mathcal{I}(G) \coloneqq \{\text{the independent sets in } G\}$. $i(G) = \#\mathcal{I}(G) = \sum_{I \in \mathcal{I}(G)} 1$.



Independence polynomial $Z_G(\lambda) \coloneqq \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$.



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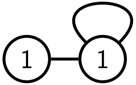

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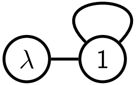
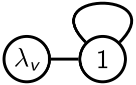


Multivariate independence polynomial



$$Z_G(\lambda_v : v \in V(G)) := \sum_{I \in \mathcal{I}(G)} \prod_{v \in I} \lambda_v.$$

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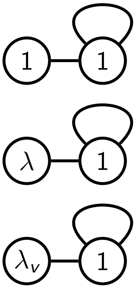
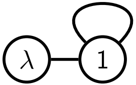


- Controls marginals, e.g., $\lambda_v \frac{\partial}{\partial \lambda_v} \log Z_G = \Pr(v \in I)$.
- Implies univariate / list coloring.

Inequalities on independence polynomials

$$Z_G(\lambda) := \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}.$$

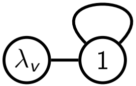


Theorem (Sah–Sawhney–Stoner–Zhao '19) For $\lambda \geq 0$,

$$\prod_{v \in V(G)} Z_{K_{d_v+1}}(\lambda)^{1/(d_v+1)} \leq Z_G(\lambda) \leq \prod_{uv \in E(G)} Z_{K_{d_u,d_v}}(\lambda)^{1/(d_u d_v)}.$$

Results

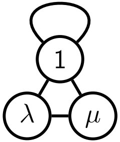
Theorem (Lee–S. '26+) $Z_G(\lambda_v : v \in V(G)) \geq \prod_{v \in V(G)} Z_{K_{d_v+1}}(\lambda_v)^{1/(d_v+1)}$.



Clique unions are still minimizers in the multivariate setting.

Results

Semiproper coloring polynomial $Z_G^{(2)}(\lambda, \mu) := \sum_{I,J \in \mathcal{I}(G), I \cap J = \emptyset} \lambda^{|I|} \mu^{|J|}$



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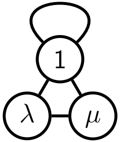
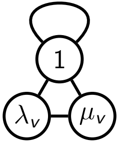


Theorem (Lee–S. '26+) $Z_G^{(2)}(\lambda_v, \mu_v : v \in V(G)) \geq \prod_{v \in V(G)} Z_{K_{d_v+1}}^{(2)}(\lambda_v, \mu_v)^{1/(d_v+1)}.$



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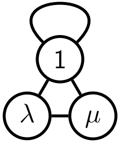

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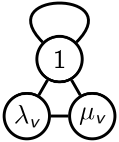

Idea: The recurrence relation reduces this to a local inequality;
Symmetrize it, then verify a low-dimensional inequality.

Strengthening: Occupancy fractions

$$\text{(Occupancy fraction)} = \frac{1}{n}\text{(expected size of an independent set)}$$

$$\alpha_G(\lambda) := \frac{1}{n}\,\mathbb{E}_{G,\lambda}|I| = \frac{\lambda}{n}\frac{\partial}{\partial\lambda}\log Z_G(\lambda)$$

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Why matters?

1. Turns local structure into global information.
2. Control existence and counting: not only $\bar{\alpha}(G)$ but also $i(G)$
3. Extremal consequences: fractional coloring, Ramsey bounds, ...

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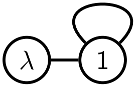
Theorem (Davies–Sandhu–S.–Tan '26++)

For $\lambda_v < 1/\Delta$, $\alpha_G(\lambda_v : v \in V(G))$ is minimized by disjoint unions of cliques.

Proof sketch

Recurrence relations

- $Z_G(\lambda) = Z_{G-w}(\lambda) + \lambda \cdot Z_{G-w-N(w)}(\lambda)$



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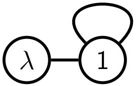
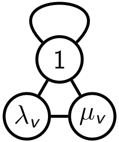
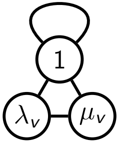


- $$Z_G^{(2)}(\lambda_v, \mu_v : v) = Z_{G-w}^{(2)}(\lambda_v, \mu_v : v) + \lambda_w \cdot Z_{G-w}^{(2)}(\lambda_v, \mu_v : v)|_{\lambda_v = 0\, \forall v \in N(w)} + \mu_w \cdot Z_{G-w}^{(2)}(\lambda_v, \mu_v : v)|_{\mu_v = 0\, \forall v \in N(w)}$$



Proof outline

Theorem (Lee–S. '26+) $Z_G^{(2)}(\lambda_v, \mu_v : v \in V(G)) \geq \prod_{v \in V(G)} Z_{K_{d_v+1}}^{(2)}(\lambda_v, \mu_v)^{1/(d_v+1)}$.



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$$+ \mu_w \cdot Z_{G-w}^{(2)}(\lambda_v, \mu_v : v)|_{\mu_v=0\, \forall v \in N(w)}$$

1. Induction on $|V(G)|$ + Recurrence at max-degree vertex w
   $\implies$ Reduce to a local inequality on w and $N(w)$.
2. Reformulate as a membership problem into a set $\mathcal{S}_\Delta$, which is log-convex.
4. Reduce to the **symmetric** case.
5. Prove the resulting low-dimensional inequality.

Proof idea

Theorem (Davies–Sandhu–S.–Tan '26++)

For $\lambda_v < 1/\Delta$, $\alpha_G(\lambda_v : v \in V(G))$ is minimized by disjoint unions of cliques.

Linear programming. Flexible framework.

Summary

Q. Which graph G has the minimum (normalized) $i(G)$?

1. Clique unions minimize the multivariate independence polynomial $Z_G(\lambda_v : v \in V(G))$ for fixed degree distribution.
2. The same phenomenon extends to semiproper coloring polynomials.
3. For small fugacities, clique unions also minimize multivariate occupancy fractions $\alpha_G(\lambda_v : v \in V(G))$.

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Q. Do these extend to graph homomorphisms to $\boldsymbol{K_q}$ **with a loop**, or more generally, **antiferromagnetic target graphs**?

Appendix

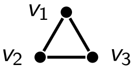
Other maximizers/minimizers

- (Cohen–Perkins–Tetali '17) $G =$  (Widom–Rowlinson model): K_{d+1} is the maximizer among the d-regular graphs.
- (Sernau '17) $\exists G$ s.t. neither $K_{d,d}$ nor K_{d+1} is a maximizer among the d-regular graphs.
- (Csikvári '16+)
  - $G =$ : the infinite d-regular tree is the minimizer among the d-regular graphs.
  - $G =$ : the infinite d-regular tree is the minimizer among the bipartite d-regular graphs.
- (Perarnau–Perkins '18)
  - $G =$ : the Petersen graph is the minimizer among the cubic graphs of girth ≥ 4.
  - $G =$ : the Heawood graph is the maximizer among the cubic graphs of girth ≥ 5.

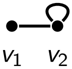
Antiferromagnetic graphs

A (edge-)weighted graph G is **antiferromagnetic** if its adjacency matrix has at most one positive eigenvalue.

$\iff \langle \mathbf{x}, G\mathbf{x}\rangle\langle \mathbf{y}, G\mathbf{y}\rangle \le \langle \mathbf{x}, G\mathbf{y}\rangle^2 \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ with $\langle \mathbf{y}, G\mathbf{y}\rangle > 0$ (**hyperbolic**)



$$\rightsquigarrow \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ v_1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 1 \\ v_3 & 1 & 1 & 0 \end{array}$$



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Conjecture (SSSZ '20) For all d-regular H and antiferromagnetic G,

$$\hom(H, G)^{1/v(H)} \le \hom(K_{d,d}, G)^{1/(2d)}.$$

Why antiferromagnetic?

- (LO**S** '25+) Support of a weighted antiferromagnetic graph $=$ blow-up of K_q or K_q°.
- (SSSZ '20) Every order 2 antiferromagnetic graph satisfies the conjecture.