Extremal questions on independent sets, colorings, *and* graph homomorphisms

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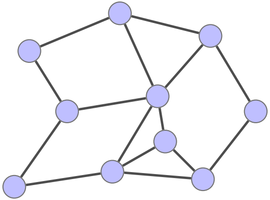
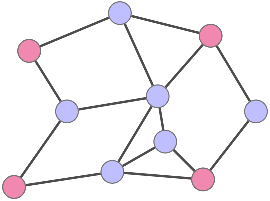
Basics

- Graph $G = (V, E)$



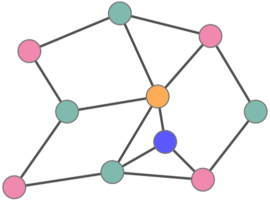
Basics

- Graph $G = (V, E)$
- Independent set



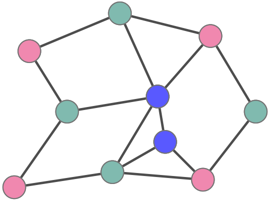
Basics

- Graph $G = (V, E)$
- Independent set
- Proper vertex coloring



Basics

- Graph $G = (V, E)$
- Independent set
- Proper vertex coloring
- Semiproper vertex coloring ("looped" color )



Questions

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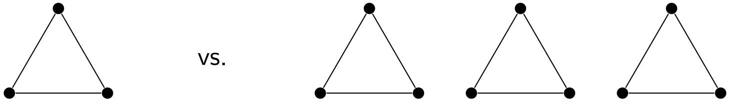
…… Among the d-regular graphs?

…… Up to appropriate normalization?



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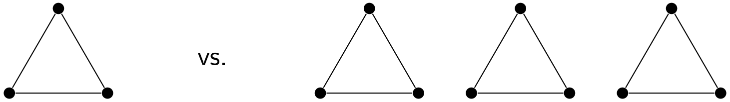
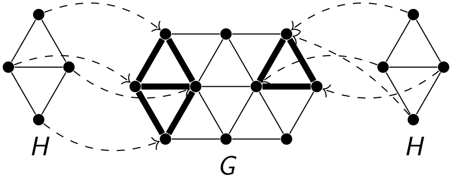


Q. Which d-regular graph H has the maximum $(\#\text{ of independent sets})^{1/v(H)}$?

Graph homomorphisms

Graph homomorphism $:=$ vertex map which preserves edges

$\mathrm{Hom}(H, G) := \{\text{the homomorphisms } H \to G\}$, $\mathrm{hom}(H, G) := |\mathrm{Hom}(H, G)|$.



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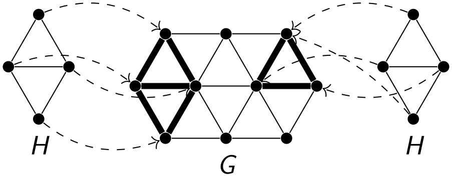
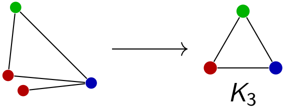
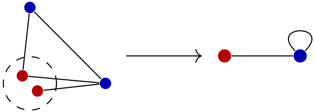


Graph homomorphisms represent several canonical objects:

- $\mathrm{Hom}(H, K_q) = \{\text{the proper } q\text{-vertex-colorings of } H\}$
- $\mathrm{Hom}(H, \bullet\!-\!\bullet^{\circlearrowleft}) = \{\text{the independent sets of } H\}$





Graph homomorphism inequalities

Q. Which d-regular graph H has the maximum/minimum (# of independent sets)$^{1/v(H)}$?

Recall $\hom(H,$  $) = (\#$ of independent sets of $H)$.

Graph homomorphism inequalities

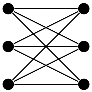
Q. Which d-regular graph H has the maximum/minimum $(\#\text{ of independent sets})^{1/v(H)}$?

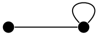
Recall $\hom(H, \bullet\!\!-\!\!\overset{\circ}{\bullet}) = (\#\text{ of independent sets of } H)$.

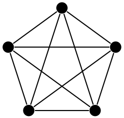
Q. Fix a graph G and a graph class $\mathcal{H}$.
Which $H \in \mathcal{H}$ has the maximum/minimum $\hom(H, G)$ up to appropriate normalization?

Maximizer Graphs

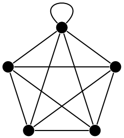
Q. Given G and $\mathcal{H} \subseteq \{\text{the } d\text{-regular graphs}\}$, which $H \in \mathcal{H}$ maximizes $\hom(H, G)^{1/v(H)}$?

G	d-regular bipartite graphs $\mathcal{B}_d$	d-regular graphs $\mathcal{H}_d$
 (independent sets)	$K_{d,d}$ (Kahn '01)	$K_{d,d}$ (Zhao '10)
K_q (proper q-colorings)	$K_{d,d}$ (Galvin–Tetali '04)	
		
General G	$K_{d,d}$ (Galvin–Tetali '04)	Not always $K_{d,d}$





K_5



K_5°

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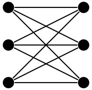
K_q° (semiproper q-colorings)		$K_{d,d}$ (SSSZ '20)
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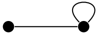

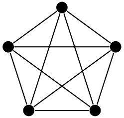

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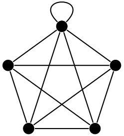

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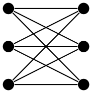
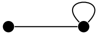
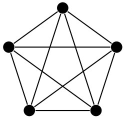
K_q° (semiproper q-colorings)		$K_{d,d}$ (SSSZ '20)
Antiferromagnetic		$K_{d,d}$?
Ferromagnetic (PSD)		K_{d+1} (SSSZ '20)
General G	$K_{d,d}$ (Galvin–Tetali '04)	Not always $K_{d,d}$





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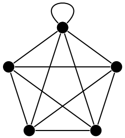
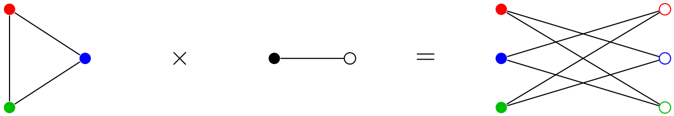


K_5°

Bipartite double cover



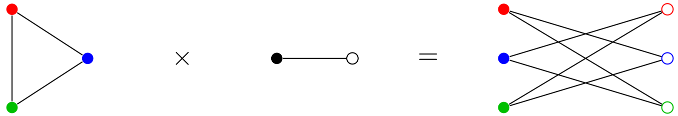
$$\boxed{\hom(H, G)^2 \le \hom(H \times K_2, G)} \qquad (\star)$$

Why important?

$$\hom(H, \bullet\!\!-\!\!\overset{\circ}{\bullet})^{1/v(H)} \le \hom(H \times K_2, \bullet\!\!-\!\!\overset{\circ}{\bullet})^{1/(2v(H))} \le \hom(K_{d,d}, \bullet\!\!-\!\!\overset{\circ}{\bullet})^{1/(2d)}$$

- (Zhao '10) $\star$ holds $\forall H$ and $G = \bullet\!\!-\!\!\overset{\circ}{\bullet}$.

Bipartite double cover



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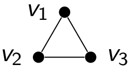
- (Zhao '10) ★ holds $\forall H$ and $G = \bullet\!-\!\overset{\circ}{\bullet}$.
- **Conjecture** (Zhao '10) ★ holds $\forall H$ and $G = K_q$.
- **Conjecture** (LOS '25+) ★ holds $\forall H$ and antiferromagnetic G.

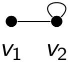
Antiferromagnetic graphs

A (edge-)weighted graph G is **antiferromagnetic** if its adjacency matrix has at most one positive eigenvalue.

$\iff \langle \mathbf{x}, G\mathbf{x}\rangle\langle \mathbf{y}, G\mathbf{y}\rangle \le \langle \mathbf{x}, G\mathbf{y}\rangle^2 \; \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ with $\langle \mathbf{y}, G\mathbf{y}\rangle > 0$ (**hyperbolic**)



$$\rightsquigarrow \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 1 \\ v_3 & 1 & 1 & 0 \end{array}$$



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Conjecture (SSSZ '20) For all d-regular H and antiferromagnetic G,

$$\hom(H, G)^{1/v(H)} \le \hom(K_{d,d}, G)^{1/(2d)}.$$

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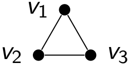

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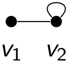

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Why antiferromagnetic?

- (LO**S** '25+) Support of a weighted antiferromagnetic graph = blow-up of K_q or K_q°.
- (SSSZ '20) Every order 2 antiferromagnetic graph satisfies the conjecture.

Lorentzian polynomials

Antiferromagnetism $\iff$ Hyperbolicity: $\langle \mathbf{x}, G\mathbf{x}\rangle\langle \mathbf{y}, G\mathbf{y}\rangle \leq \langle \mathbf{x}, G\mathbf{y}\rangle^2$

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Hyperbolicity $\rightsquigarrow$ Negative correlation:

E.g., $A \subseteq V(G)$, $\mathbf{x} = \mathbf{1}_A$, $\mathbf{y} = \mathbf{1}$. Sample $(i,j) \in V(G)^2$ proportional to G_{ij}. Then

$$\langle \mathbf{x}, G\mathbf{x}\rangle\langle \mathbf{y}, G\mathbf{y}\rangle \le \langle \mathbf{x}, G\mathbf{y}\rangle^2 \iff \Pr(i,j \in A) \le \Pr(i \in A)\Pr(j \in A).$$

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The theory of **Lorentzian polynomials** is a powerful framework to describe negative correlation.

Applying Lorentzian polynomials

G-chromatic function: for $V(G) = [n]$,

$$h_H(x_1, \ldots, x_n; G) := \sum_{\phi:\ V(H) \to V(G)} \prod_{uv \in E(H)} G(\phi(u), \phi(v)) \prod_{v \in V(H)} x_{\phi(v)}.$$

In particular, $h_H(\mathbf{1}; G) = \hom(H, G)$.

Theorem (Lee–Oh–S. '25+) G antiferromagnetic $\implies$ $h_{K_t}(\,\cdot\,; G)$ Lorentzian.

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Theorem (Lee–Oh–S. '25+) G antiferromagnetic $\implies$ $h_{K_t}(\,\cdot\,; G)$ Lorentzian.

Theorem (Brändén–Huh '20) Lorentzian polynomials $\implies$ Alexandrov–Fenchel / log-concavity inequalities.

$\implies$ K_t satisfies a strengthening of $\hom(H, G)^2 \le \hom(H \times K_2, G)$.

$\implies$ $\boxed{\text{For the } H\text{'s we build, } \hom(H, G)^2 \le \hom(H \times K_2, G).}$

Results on maximizers

Conjecture

1. (Zhao '11) For all H and q, $\hom(H, K_q)^2 \leq \hom(H \times K_2, K_q)$.
2. (LO**S** '25+) For all H and antiferromagnetic G, $\hom(H, G)^2 \leq \hom(H \times K_2, G)$.

Theorem (LO**S** '25+) Verified the conjectures for two new families:

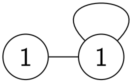
1. every $H \in \mathcal{H}_1$ satisfies the conjecture of (Zhao '11);
2. every $H \in \mathcal{H}_2$ satisfies the conjecture of (LO**S** '25+). **($\Longleftarrow$ Lorentzian polynomials)**

Importance: First application of the theory of Lorentzian polynomials in extremal combinatorics.

Minimizer Graphs

Independent polynomials

$\mathcal{I}(H) :=$ {the independent sets in H}. $\#\mathcal{I}(H) = \sum_{I \in \mathcal{I}(H)} 1.$



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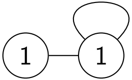
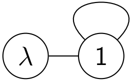


Independence polynomial $Z_H(\lambda) := \sum_{I \in \mathcal{I}(H)} \lambda^{|I|}$.



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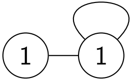

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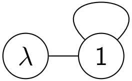
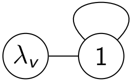


Multivariate independence polynomial



$$Z_H(\lambda_v : v \in V(H)) := \sum_{I \in \mathcal{I}(H)} \prod_{v \in I} \lambda_v.$$

Equal to $Z_H(\lambda)$ if $\lambda_v = \lambda \ \forall v \in V(H)$.

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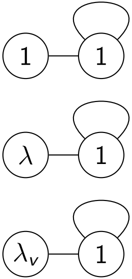

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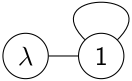
Equal to $Z_H(\lambda)$ if $\lambda_v = \lambda \ \forall v \in V(H)$.

Why multivariate?

- Controls marginals, e.g., $\frac{\partial}{\partial \lambda_v} Z_H = \Pr(v \in I)$.
- Implies univariate / list coloring.

Inequalities on independence polynomials

$$Z_H(\lambda) = \sum_{I \in \mathcal{I}(H)} \lambda^{|I|}.$$

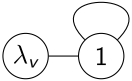


Assume $\lambda \geq 0$.

- (SSSZ '19) $Z_H(\lambda) \geq \prod_{v \in V(H)} Z_{K_{d_v+1}}(\lambda)^{1/(d_v+1)}$.
- (SSSZ '20) $Z_H(\lambda) \leq \prod_{uv \in E(H)} Z_{K_{d_u,d_v}}(\lambda)^{1/(d_u d_v)}$.

Results on minimizers

Theorem (Lee–S. '26+) $Z_H(\lambda_v : v \in V(H)) \geq \prod_{v \in V(H)} Z_{K_{d_v+1}}(\lambda_v)^{1/(d_v+1)}$.



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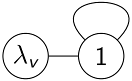
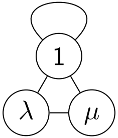


Semiproper coloring polynomial $Z_H^{(2)}(\lambda, \mu) := \sum_{I,J \in \mathcal{I}(H), I \cap J = \emptyset} \lambda^{|I|} \mu^{|J|}$



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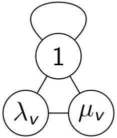

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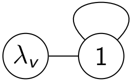

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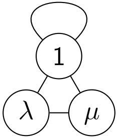

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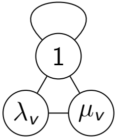
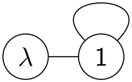


Conjecture Analogous results hold for all antiferromagnetic target graphs.

Recurrence relations

- $Z_H(\lambda) = Z_{H-w}(\lambda) + \lambda \cdot Z_{H-w-N(w)}(\lambda)$



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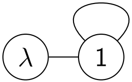
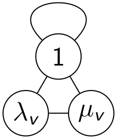
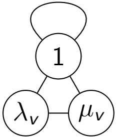


- $$Z_H^{(2)}(\lambda_v, \mu_v : v) = Z_{H-w}^{(2)}(\lambda_v, \mu_v : v) + \lambda_w \cdot Z_{H-w}^{(2)}(\lambda_v, \mu_v : v)|_{\lambda_v=0\,\forall v\in N(w)} + \mu_w \cdot Z_{H-w}^{(2)}(\lambda_v, \mu_v : v)|_{\mu_v=0\,\forall v\in N(w)}$$



Proof outline



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1. Induction on $|V(H)|$ + Recurrence at max-degree vertex w
   $\implies$ Reduce to a local inequality on w and $N(w)$.
2. ★ Reformulate as a membership problem into a set $\mathcal{S}_\Delta$, which is log-convex.
3. Reduce to the **symmetric** case.
4. Prove the resulting low-dimensional inequality.

Google DeepMind

February 11, 2026 Research

Accelerating Mathematical and Scientific Discovery with Gemini Deep Think

Thang Luong and Vahab Mirrokni

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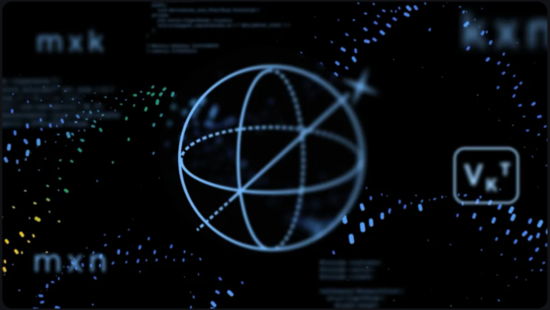

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Under direction from expert mathematicians and scientists, Gemini Deep Think

For research-level math, Aletheia has already enabled several advancements, produced via varying levels of autonomous research:

- Reliable autonomous research. A research paper (Feng26) generated by AI without any human intervention, which calculates certain structure constants in arithmetic geometry called eigenweights.
- AI-guided collaboration. A research paper (LeeSeo26) demonstrating human-AI collaboration in proving bounds on systems of interacting particles called independent sets.
- An extensive semi-autonomous evaluation (Feng et al., 2026b) of 700 open problems on Bloom's Erdős Conjectures database, including autonomous solutions to four open questions listed there. On Erdős-1051, our model autonomously solved and helped lead to a generalization reported in a research paper (BKKKZ26).

The agent also contributed intermediate propositions on two further papers, (FYZ26) and (ACGKMP26). It is also of note that there has been prior work using Gemini for research-level math at a smaller scale in terms of collaborations and the number of problems tackled.

Following extensive discussions with the mathematical community, we suggest a taxonomy to classify AI-assisted mathematics research by significance and degree of AI contribution - contributing to the wider discussion on responsible documentation, evaluation and communication of AI-generated results. Level 2 ("publishable quality") works have been submitted to reputable journals. Currently, we do not claim any Level 3 ("Major Advance") and Level 4 ("Landmark Breakthrough") results.

	Human with secondary AI input	Human-AI Collaboration	Essentially Autonomous
Level 0: Negligible Novelty			Erdős-652, 654, 1040 (Feng et al., 2026b)
Level 1: Minor Novelty			Erdős-1051 (Feng et al., 2026b)
Level 2: Publishable Research*	Complexity Bounds (ACGKMP26), Arithmetic Volumes (FYZ26)	Independence Polynomials (LeeSeo26), Generalized Erdős-1051 (BKKKZ26)	Eigenweights (Feng26)
Level 3: Significant Advance			
Level 4: Landmark Breakthrough			

Classification of all AI-assisted mathematics results encompassed in this work. *Works listed as Level 2 in this table have been submitted for publications.

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Summary

Q. Fix a graph G and a graph class $\mathcal{H}$.
Which $H \in \mathcal{H}$ has the maximum/minimum $\hom(H, G)$ up to appropriate normalization?

Let G be antiferromagnetic.

- (Lee–Oh–S. '25+) Towards the conjecture "$K_{d,d}$ is the maximizer" by analyzing

$$\hom(H, G)^2 \leq \hom(H \times K_2, G).$$

- (Lee–S. '26+) Towards the conjecture "K_{d+1} is the minimizer" by consider multivariate versions for $G = K_2^\circ$ and K_3°.

Summary …… and **Thank you!**

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Appendix

Lorentzian polynomials

Theorem (Lee–Oh–S. '25+) If H is a clique and G is antiferromagnetic,

$$h_H(x_1, \dots, x_n; G) := \sum_{\phi:\ V(H)\to V(G)} \prod_{uv\in E(H)} G(\phi(u), \phi(v)) \prod_{v\in V(H)} x_{\phi(v)}$$

is Lorentzian.

Theorem (Brändén–Huh) Let $\mathbf{x} = (x_1, \dots, x_n)$ and $f(\mathbf{x})$: a homogeneous polynomial of deg d.

$$V(\mathbf{x}_1, \dots, \mathbf{x}_d) := \frac{1}{d!}\partial_{\lambda_1} \dots \partial_{\lambda_d} f(\lambda_1\mathbf{x}_1 + \cdots + \lambda_d\mathbf{x}_d).$$

If f is Lorentzian, then for all $\mathbf{a}_1 \in \mathbb{R}^n$ and $\mathbf{a}_2, \dots, \mathbf{a}_d \in (\mathbb{R}_{\geq 0})^n$,

$$V(\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_3, \dots, \mathbf{a}_d) \cdot V(\mathbf{a}_2, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_d) \leq V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_d)^2.$$

Set $H = K_d$ and $f(\mathbf{x}) = h_H(\mathbf{x}; G)$.

- $V(\mathbf{1}_{A_1}, \dots, \mathbf{1}_{A_d})$: the number of "list colorings" of K_d
- $V(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}) \cdot V(\mathbf{b}, \mathbf{b}, \dots, \mathbf{b}) \leq V(\mathbf{b}, \mathbf{a}, \dots, \mathbf{a}) \cdot V(\mathbf{a}, \mathbf{b}, \dots, \mathbf{b})$

Irregular graphs

If H is irregular, the condition becomes

$$\hom(H, G) \leq \prod_{uv \in E(H)} \hom(K_{d_u, d_v}, G)^{1/(d_u d_v)}.$$

If H is d-regular, this is equivalent to $\hom(H, G) \leq \hom(K_{d,d}, G)^{e(H)/d^2}$, or

$$\hom(H, G)^{1/v(H)} \leq \hom(K_{d,d}, G)^{1/(2d)}.$$

Other maximizers/minimizers

- (Cohen–Perkins–Tetali '17) $G =$  (Widom–Rowlinson model): K_{d+1} is the maximizer among the d-regular graphs.
- (Sernau '17) $\exists G$ s.t. neither $K_{d,d}$ nor K_{d+1} is a maximizer among the d-regular graphs.
- (Csikvári '16+)
  - $G =$ : the infinite d-regular tree is the minimizer among the d-regular graphs.
  - $G =$ : the infinite d-regular tree is the minimizer among the bipartite d-regular graphs.
- (Perarnau–Perkins '18)
  - $G =$ : the Petersen graph is the minimizer among the cubic graphs of girth ≥ 4.
  - $G =$ : the Heawood graph is the maximizer among the cubic graphs of girth ≥ 5.