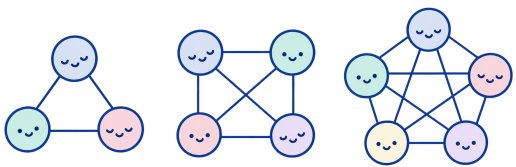
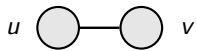
Antiferromagnetic models are clique-minimizing

Jaehyeon Seo (Yonsei University)
Joint work with Joonkyung Lee (KIAS)

2026 KMS Annual Meeting
October 31, 2026

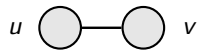


Warm-up: Two lists on an edge



Color an edge uv **properly**: the two colors must be different.

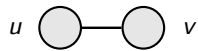
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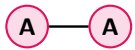
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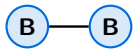
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Example: $A = \{1, 2\},\ B = \{2, 3\}$.



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$\#(BB) = 2$

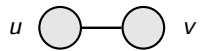


$\#(AB) = 3$



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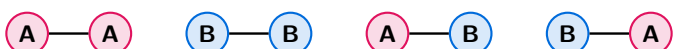
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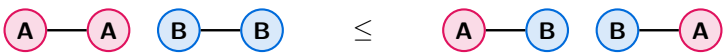


$\#(AA) = 2$ $\#(BB) = 2$ $\#(AB) = 3$ $\#(BA) = 3$

$$\#(AA)\,\#(BB) = 4 \leq 9 = \#(AB)\,\#(BA).$$

Warm-up: Repulsion on one edge

For every pair of lists $A, B \subseteq [q]$,

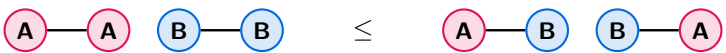


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Q. What does this **one-edge** inequality imply for larger graphs?

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For a general matrix H, $(*)$ defines $\hom(G, H)$, the **number of H-colorings** of G.

H-colorings

H: a symmetric, entrywise nonnegative $q \times q$ matrix.

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Main question

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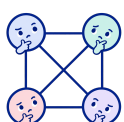
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Q. What makes both matrices (or coloring models) clique-minimizing?

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($H = I$ gives $(x_1^2 + \cdots + x_q^2)(y_1^2 + \cdots + y_q^2) \ge (x_1 y_1 + \cdots + x_q y_q)^2$, the usual CS.)

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H: symmetric, entrywise nonnegative.

Definition H is **antiferromagnetic** if

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Indeed, an analogous inequality holds for all general G:

$$\hom(G,H) \le \prod_{v\in V(G)} \hom(K_{d_v+1},H)^{1/(d_v+1)}.$$

(d_v: degree of v)

Cliques as extremizers

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Reverse Cauchy–Schwarz on one edge suggests reversing this inequality.

Conjecture (Lee–S. '26a+) Every **antiferromagnetic** H is **clique-minimizing**:

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Theorem (Lee–S. '26b+)

clique-minimizing with arbitrary vertex weights $\iff$ antiferromagnetic

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We use the properties of Lorentzian polynomials to prove **entropy inequalities**.

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Appendix

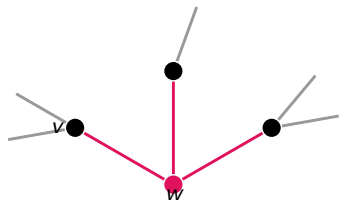
Appendix

1. Strategy and Lorentzian polynomials
2. Entropy: climbing clique sizes
3. A local inequality for independent sets
4. Ferromagnetic matrices and Cauchy–Schwarz

Strategy

Induct on $|V(G)|$:

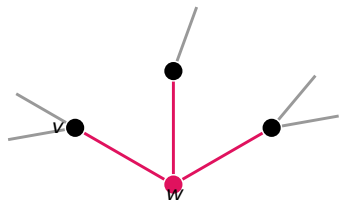
1. Peel off a vertex w of maximum degree Δ and fix its color α.
2. Neighbors' weights get multiplied by $\boldsymbol{H}_\alpha$.
3. Log-convexity $\rightsquigarrow$ one **local inequality** per neighbor v, of degree $d \leq \Delta$.



Strategy

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Two cases:

- $d = \Delta$: **concavity** (Lorentzian polynomials);
- $d < \Delta$: **climbing** from K_{d+1} to $K_{\Delta+1}$ (entropy).

Lorentzian polynomials

Definition (Brändén–Huh '20; Anari–Liu–Oveis Gharan–Vinzant) A homogeneous $f \geq 0$ of degree d is **Lorentzian** if

- its support is M-convex, and
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Examples:

- $e_k(\mathbf{x})$: the elementary symmetric polynomial;
- basis generating polynomials of matroids (Mason's conjecture).

Basic properties

If f is Lorentzian of degree d:

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2. $f^{1/d}$ is **concave** on $\mathbb{R}^n_{>0}$, which gives the tangent-plane inequality

$$f^{1/d}(\mathbf{z}) \leq \nabla f^{1/d}(\mathbf{x}) \cdot \mathbf{z}.$$

The clique polynomial

Lemma (Lee–Oh–S.) If H is antiferromagnetic, then

$$Z_m^H(\mathbf{x}) \coloneqq \sum_{c\colon [m]\to[q]} \prod_{i<j} H_{c(i)c(j)} \prod_{i=1}^{m} x_{c(i)}$$

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- Induct down to $Z_2^H = \mathbf{x}^\top H \mathbf{x}$.
- Example: for $H = J - I$ $(n \times n)$, $Z_k^H = k!\, e_k(\mathbf{x})$.
- Concavity of $(Z_{\Delta+1}^H)^{1/(\Delta+1)}$ gives the case $d = \Delta$.

Shearer's inequality, sharpened

Lemma (Shearer '86; Han '78) For any random vector $\mathbf{X} = (X_1, \ldots, X_m)$,

$$\sum_{j=1}^{m} \mathbb{H}(\mathbf{X}_{[m]\setminus j}) \geq (m-1)\,\mathbb{H}(\mathbf{X}).$$

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Lemma (refined Shearer; Lee–S.) If $\mathbf{X}$ is exchangeable and $\mathbb{E}\ \mathbf{x}^{\mathbf{T}(\mathbf{X})}$ is Lorentzian,

$$\sum_{j=1}^{m} \mathbb{H}(\mathbf{X}_{[m]\setminus j}) \geq (m-2)\,\mathbb{H}(\mathbf{X}) + \sum_{j=1}^{m} \mathbb{H}(X_j).$$

- Gain: the total correlation $\sum_j \mathbb{H}(X_j) - \mathbb{H}(\mathbf{X}) \geq 0$.
- Example: a uniformly random proper coloring of K_m.

Engine: one coordinate carries a $1/m$ share

Lemma Let μ, ν be exchangeable laws on $[q]^m$. If ν is Lorentzian, then

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- A multiset analogue of **entropic independence** (Anari–Jain–Koehler–Pham–Vuong).

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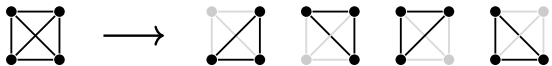
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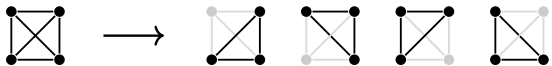
Under repulsion, one coordinate carries only a $1/m$ share of the information.

Climbing clique sizes



K_m m copies of K_{m-1}; each edge survives in $m-2$

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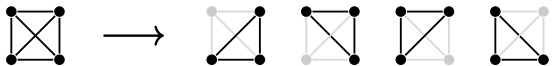


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Gibbs variational principle: the normalized clique count becomes an optimized pressure

$$F_m = \sup_{\mu}\Big\{\mathbb{E}_\mu\, U_m + \mathbb{H}(\mathbf{X}) - \sum_j \mathbb{H}(X_j)\Big\} \quad (\mu \text{ exchangeable Lorentzian}).$$

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- Energy: $\sum_j U_{m-1}(\mathbf{X}_{-j}) = (m-2)\, U_m(\mathbf{X})$. Entropy: refined Shearer.
- Together: $m\, F_{m-1} \geq (m-2)\, F_m$.
  $F_{d+1} \leq 0$ (concavity) $\Rightarrow F_{d+2} \leq 0 \Rightarrow \cdots \Rightarrow F_{\Delta+1} \leq 0$: the case $d < \Delta$.

Recurrence for the local inequality

For $H = H_{\mathrm{ind}}$ with vertex weights λ_v on "occupied", the recurrence reduces the theorem to the following local inequality.

For $\Delta \geq 1$, $1 \leq d_1, \ldots, d_\Delta \leq \Delta$, and $\lambda_0, \lambda_1, \ldots, \lambda_\Delta \geq 0$,

$$\prod_{i=1}^{\Delta}(1 + d_i\lambda_i)^{1/d_i} + \lambda_0 \geq (1 + (\Delta + 1)\lambda_0)^{1/(\Delta+1)} \prod_{i=1}^{\Delta}(1 + (d_i + 1)\lambda_i)^{1/(d_i+1)}.$$

Ferromagnetic matrices

Theorem (Sah–Sawhney–Stoner–Zhao '20) If $H \geq 0$ is positive semidefinite, then for every graph G,

$$\hom(G, H) \leq \prod_{v \in V(G)} \hom(K_{d_v+1}, H)^{1/(d_v+1)}.$$

- The same with vertex weights.
- One edge: $(\mathbf{x}^\top H \mathbf{y})^2 \leq (\mathbf{x}^\top H \mathbf{x})(\mathbf{y}^\top H \mathbf{y})$ $\forall \mathbf{x}, \mathbf{y} \geq 0 \iff H$ is PSD.
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  So, with vertex weights: clique-maximizing $\iff$ PSD.
- A 0-1 PSD matrix with no zero rows has all-ones diagonal blocks of sizes s_k:

$$\hom(G, H) = \prod_{\text{components } C} \sum_k s_k^{|C|}.$$

  Nontrivial examples have other entries, e.g. the ferromagnetic Ising model.

Cauchy–Schwarz and positive semidefiniteness

Throughout, H is symmetric and **entrywise nonnegative**.

Fact H is PSD if and only if

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Proof of the converse. Write any $\mathbf{z} \in \mathbb{R}^q$ as $\mathbf{z} = \mathbf{x} - \mathbf{y}$, with $\mathbf{x},\mathbf{y} \ge 0$. Entrywise nonnegativity and Cauchy–Schwarz give

$$\begin{aligned}\mathbf{z}^\top H\mathbf{z} &= \mathbf{x}^\top H\mathbf{x} + \mathbf{y}^\top H\mathbf{y} - 2\mathbf{x}^\top H\mathbf{y} \\ &\ge \left(\sqrt{\mathbf{x}^\top H\mathbf{x}} - \sqrt{\mathbf{y}^\top H\mathbf{y}}\right)^2 \ge 0.\end{aligned}$$

The forward implication is the usual Cauchy–Schwarz inequality for a PSD form.

Reverse Cauchy–Schwarz and the spectrum

Let $H \neq 0$ be symmetric and entrywise nonnegative, and put $Q(\mathbf{x}) = \mathbf{x}^\top H \mathbf{x}$.

- If H has at most one positive eigenvalue, its restriction to $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$, for independent $\mathbf{x}, \mathbf{y} > 0$, has determinant at most zero:

$$Q(\mathbf{x})Q(\mathbf{y}) - (\mathbf{x}^\top H \mathbf{y})^2 \leq 0.$$

- Conversely, fix $\mathbf{x} > 0$. For $\mathbf{x}^\top H \mathbf{z} = 0$, apply reverse Cauchy–Schwarz to $\mathbf{x}$ and $\mathbf{x} + t\mathbf{z} > 0$, with $t \neq 0$ sufficiently small:

$$Q(\mathbf{x})\big(Q(\mathbf{x}) + t^2 Q(\mathbf{z})\big) \leq Q(\mathbf{x})^2, \qquad \text{so } Q(\mathbf{z}) \leq 0.$$

Thus Q is nonpositive on a codimension-one subspace. Dependent vectors give equality; boundary vectors follow by continuity.