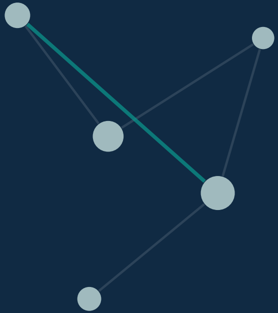
ASIACOMB 2026 / Contributed talk

Degree-sequence bounds for independent sets

via multivariate local occupancy



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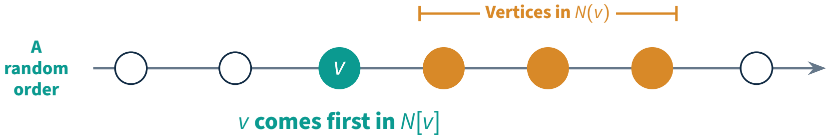
Caro–Wei theorem

Bound from max degree

$$\alpha(G) \geq \frac{n}{\chi(G)} \geq \frac{n}{\Delta + 1}$$

Caro–Wei

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d_v + 1}$$



Caro (1979); Wei (1981).

Triangle-free graphs

$$\alpha(G) \geq (1 - o(1))\frac{n \log d}{d} \quad \text{for triangle-free } G \quad \Longrightarrow \quad R(3,k) \leq (1 + o(1))\frac{k^2}{\log k}$$

1980–81	1983	1991	2017–18
AKS	**Shearer**	**Shearer**	**DJPR**
$\Omega(n \log d / d)$	average degree	degree sequence	occupancy method

Ajtai–Komlós–Szemerédi (1980/81); Shearer (1983/91); Davies–Jenssen–Perkins–Roberts (2017/18).

Hard-core model

$$\boldsymbol{\lambda} = (\lambda_v : v \in V(G)), \qquad \mathbb{P}_{G,\boldsymbol{\lambda}}(X = I) := \frac{1}{Z_G(\boldsymbol{\lambda})} \prod_{v \in I} \lambda_v$$

Small λ

mostly small sets

$\lambda = 1$

uniform over all sets

Large λ

near maximum sets

$$\alpha(G) \geq \mathbb{E}_{G,\boldsymbol{\lambda}}|X| = \sum_{v \in V(G)} p_v, \qquad p_v := \mathbb{P}(v \in X).$$

Occupancy conjecture

Conjecture (Davies–Kang)

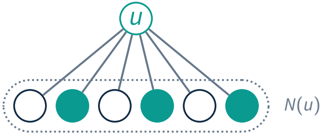
$$\mathbb{E}_{G,\lambda\mathbf{1}}|X| \overset{?}{\geq} \sum_{v \in V(G)} \frac{\lambda}{1+(d_v+1)\lambda} \qquad (\lambda > 0)$$

hard-core occupancy (finite λ) $\xrightarrow{\lambda \to \infty}$ maximum independent sets $\longrightarrow$ Caro–Wei

Davies–Kang (2025), Conjecture A.

Local constraint, $\lambda = 1$

X is uniformly random in $\mathcal{I}(G)$



$Y = |N(u) \cap X|$

$$\mathbb{E}\, Y = \sum_{v \in N(u)} p_v$$

For each $u \in V(G)$:

$$p_u = \mathbb{P}(u \in X) = \frac{1}{2}\,\mathbb{P}(Y = 0)$$

$$\mathbb{E}\, Y \geq \mathbb{P}(Y \geq 1) = 1 - \mathbb{P}(Y = 0)$$

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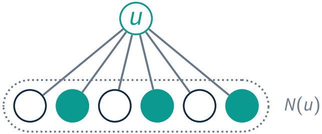

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$$\boxed{2p_u + \sum_{v \in N(u)} p_v \geq 1}$$

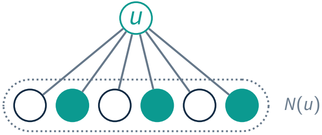
$\xrightarrow[d_v \leq \Delta]{\textbf{sum over } u}$

Immediate consequence

$$\mathbb{E}_{G,\mathbf{1}} |X| \geq \frac{n}{\Delta + 2}$$

Local constraint, general λ

X follows the hard-core model with positive fugacity vector λ



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For each $u \in V(G)$:

$$p_u = \mathbb{P}(u \in X) = \frac{\lambda_u}{1+\lambda_u}\,\mathbb{P}(Y = 0)$$

$$\mathbb{E}\, Y \geq \mathbb{P}(Y \geq 1) = 1 - \mathbb{P}(Y = 0)$$

$$\left(1 + \frac{1}{\lambda_u}\right) p_u + \sum_{v \in N(u)} p_v \geq 1$$

sum over u

$d_v \leq \Delta$

$\lambda_v := \lambda$

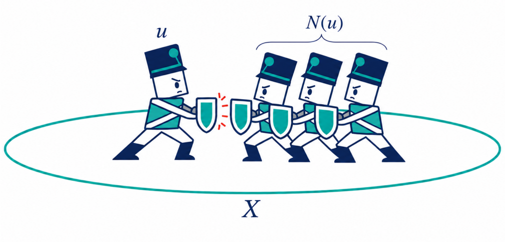
Immediate consequence

$$\mathbb{E}_{G,\lambda \mathbf{1}} |X| \geq \frac{n}{\Delta + 1 + 1/\lambda}$$

Local occupancy

For $\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda} \in [0, \infty)^{V(G)}$, G has *local* $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$*-occupancy* if $\forall u$,

$$\beta_u \, \mathbb{P}_{G,\boldsymbol{\lambda}}(u \in X) + \gamma_u \sum_{v \in N_G(u)} \mathbb{P}_{G,\boldsymbol{\lambda}}(v \in X) \geq 1.$$



A weighted balance:

u versus $N(u)$.

Davies–Jenssen–Perkins–Roberts (2017/18); Davies–Kang–Pirot–Sereni (2020+); Davies–Sandhu–Seo–Tan (2026+).

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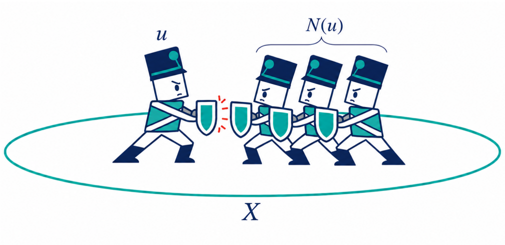


A weighted balance:
u versus $N(u)$.

Graphs with specific properties → Local $(\beta, \gamma, \boldsymbol{\lambda})$-occupancy → Desired results

Davies–Jenssen–Perkins–Roberts (2017/18); Davies–Kang–Pirot–Sereni (2020+); Davies–Sandhu–Seo–Tan (2026+).

Theorem 1

Occupancy conjecture, general **multivariate, restricted λ**

Let G have maximum degree Δ, and let $0 \leq \lambda_v < 1/\Delta$ for every v. Then

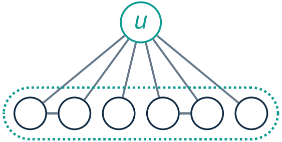
$$\mathbb{E}_{G,\boldsymbol{\lambda}}|X| \geq \sum_{v \in V(G)} \frac{\lambda_v}{1 + (d_v + 1)\lambda_v}.$$

Davies–Sandhu–Seo–Tan (2026+), Theorem 1.

Theorem 2

Degree-sequence bound, locally sparse

univariate, restricted λ



sparse $G[N(u)]$

G has *bounded local max avg deg*:
$\mathrm{mad}(G[N(u)]) \leq b \quad \forall u.$

$b = 0$: triangle-free

Approximate form

If $0 < \lambda \lesssim_b 1/\Delta$, then

$$\mathbb{E}_{G,\lambda\mathbf{1}}|X| \gtrsim_b \sum_{v \in V(G)} \frac{\log(1 + d_v\lambda)}{d_v}.$$

Davies–Sandhu–Seo–Tan (2026+), Theorem 2.

What changes

General	univariate $o(\Delta^{-2})$	$\Longrightarrow$	multivariate $< \ 1/\Delta$
Triangle-free	degree sequence $o(\Delta^{-4})$	$\Longrightarrow$	degree sequence $0.109597/\Delta$
Method	induction Taylor estimates	$\Longrightarrow$	spectral LP no induction

Local constraints to LP

$$\beta_u p_u + \gamma_u \sum_{v \in N(u)} p_v \geq 1 \quad \forall u$$

Local constraints to LP

$$\beta_u p_u + \gamma_u \sum_{v \in N(u)} p_v \geq 1 \quad \forall u$$

$A =$ adjacency matrix
$B = \mathrm{diag}(\beta), \quad \Gamma = \mathrm{diag}(\gamma)$

Primal LP

$$\begin{aligned} \text{minimize} \quad & \mathbf{1}^\top x \\ \text{subject to} \quad & (B + \Gamma A)x \geq \mathbf{1}, \\ & x \geq 0 \end{aligned}$$

the marginal vector p is feasible

Dual LP

$$\begin{aligned} \text{maximize} \quad & \mathbf{1}^\top y \\ \text{subject to} \quad & (B + A\Gamma)y \leq \mathbf{1}, \\ & y \geq 0 \end{aligned}$$

construct one feasible y

Explicit dual certificate

Under λ small, $\quad y^\star = (B + A\Gamma)^{-1}\mathbf{1} \geq 0 \quad \Longrightarrow \quad \mathbf{1}^\top p \geq \mathbf{1}^\top y^\star$

Neumann series

If T is a real square matrix with $\underbrace{\rho(T)}_{\text{spectral radius of } T} < 1$, then $(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$.

Let $D := \operatorname{diag}(d_v)$, $L := D - A$, and $H := B + D\Gamma$, and set

$$T := L\Gamma H^{-1}, \qquad B + A\Gamma = (I - T)H.$$

Neumann series

Using $B + A\Gamma = (I - T)H$ and $T = L\Gamma H^{-1}$.

$$\mathbf{1}^\top y^\star = \mathbf{1}^\top (B + A\Gamma)^{-1}\mathbf{1} = \mathbf{1}^\top H^{-1}(I - T)^{-1}\mathbf{1}$$
$$= \sum_{k=0}^{\infty} \mathbf{1}^\top H^{-1} T^k \mathbf{1} =: \sum_{k=0}^{\infty} c_k.$$

Goal

$$\sum_{k=0}^{\infty} c_k \geq c_0 = \sum_v \frac{1}{\beta_v + d_v\gamma_v}$$

Theorem 1

$$c_k \geq 0 \quad \forall k \geq 1$$

general G

Theorem 2

$$c_1 \geq \left|\textstyle\sum_{k\geq 2} c_k\right|$$

locally sparse G

Takeaway

Occupancy conjecture $\mathbb{E}_{G,\lambda\mathbf{1}}|X| \overset{?}{\geq} \sum_{v\in V(G)} \frac{\lambda}{1+(d_v+1)\lambda}$ $(\forall G,\ \lambda > 0)$

Our method

(Multivariate) local occupancy + spectral LP

Theorem 1
arbitrary graphs, multivariate
$0 \ \leq\ \lambda_v\ <\ 1/\Delta$

Theorem 2
locally sparse, univariate
$0\ <\ \lambda\ \lesssim_b\ 1/\Delta$

For arbitrarily large $\boldsymbol{\lambda}$, the LP must be strengthened.

Full parameters

For Theorem 2, let

$$\begin{aligned} D_v &:= d_v(1+\lambda)^b \log(1+\lambda), \\ \beta_v &:= \frac{1+\lambda}{\lambda} \frac{D_v}{W(D_v)(1+W(D_v))}, \\ \gamma_v &:= \frac{1+\lambda}{\lambda} \frac{D_v}{d_v(1+W(D_v))}. \end{aligned}$$

Then

$$\mathbb{E}|X| \geq \sum_v \frac{1}{\beta_v + d_v\gamma_v} = \frac{\lambda}{1+\lambda} \sum_v \frac{W(D_v)}{D_v}.$$

Dual conditions

The certificate $y^\star = (B + A\Gamma)^{-1}\mathbf{1}$ is feasible when

$$\beta_v, \gamma_v, \lambda_v > 0, \qquad d_v \frac{\gamma_v}{\beta_v} < 1, \qquad \sum_{u \in N(v)} \frac{\gamma_u}{\beta_u} \le 1.$$

These conditions also give

$$\rho(L\Gamma H^{-1}) < 1,$$

so the Neumann series used in the proof converges.

Counting corollary

For triangle-free G, integration gives, for $0 < \lambda < 0.109597/\Delta$,

$$\log Z_G(\lambda) \geq \sum_{v \in V(G)} \frac{1}{2d_v} \left[W\big(d_v \log(1+\lambda)\big)^2 + 2W\big(d_v \log(1+\lambda)\big) \right].$$

Degree-sequence lower bound for $\log Z_G$

References

Degree sequence and triangle-free history

- Caro (1979); Wei (1981)
- Ajtai–Komlós–Szemerédi (1980/81)
- Shearer (1983, 1991)
- Kelly–Postle (2024)
- Martinsson–Steiner (2025)

Hard-core and occupancy

- Davies–Jenssen–Perkins–Roberts (2017, 2018)
- Sah–Sawhney–Stoner–Zhao (2019)
- Davies et al. (2020, 2021)
- Davies–Kang (2025)
- Davies–Sandhu–Tan (2025)
- Lee–Seo (2026)