What counts do cliques minimize?

Independent Sets, colorings, and more

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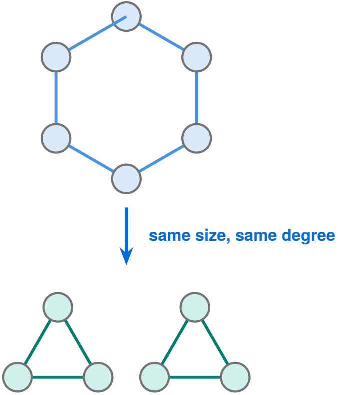
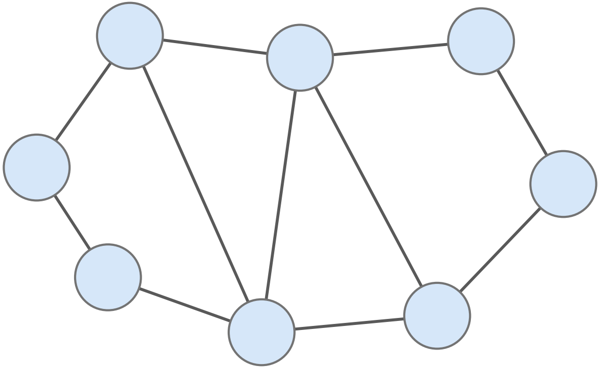
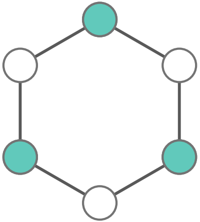


Graphs



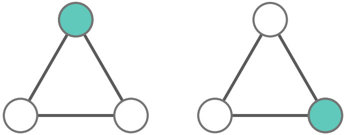
Independent sets

Independent set: no two selected vertices are adjacent; $i(G)$ counts them.



cycle C_6
$i(C_6) = \mathbf{18}$

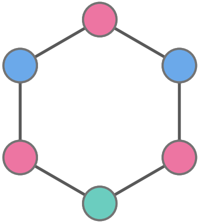
$>$



two cliques $2K_3$
$i(2K_3) = \mathbf{16}$

Proper colorings

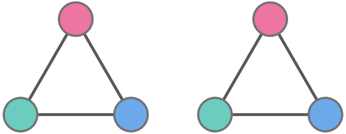
Proper 3-coloring: adjacent vertices receive different colors; $P_G(3)$ counts them.



cycle C_6

$$P_{C_6}(3) = \mathbf{66}$$

$>$



two cliques $2K_3$

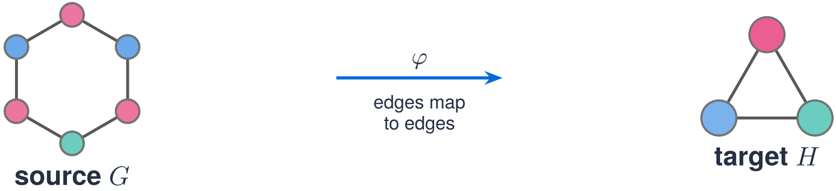
$$P_{2K_3}(3) = \mathbf{36}$$

Why do the same cliques give the smaller count in both examples?

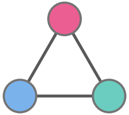
Graph homomorphisms: one language

Graph homomorphism: a vertex map $\varphi : V(G) \to V(H)$ that preserves edges.

$$\hom(G, H) := \#\{\text{graph homomorphisms } G \to H\}.$$

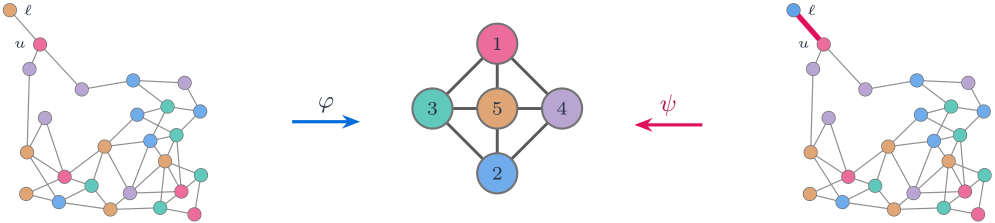


Independent sets
$H = K_2^\circ$



Proper colorings
$H = K_q$

Graph homomorphisms: one language



source G

homomorphism

target H

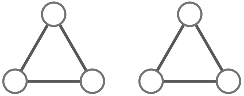
$K_5 - \{12, 34\}$

same source G

not a homomorphism

$u\ell \mapsto 12 \notin E(H).$

Graph homomorphisms: one language



$G_1 \sqcup G_2$

$$\hom(G_1 \sqcup G_2, H) = \hom(G_1, H)\ \hom(G_2, H).$$

Multiplicativity suggests comparing *per vertex*:

$$\hom(G, H)^{1/v(G)}.$$

d-regular source

When does K_{d+1} minimize

$$\hom(G, H)^{1/v(G)}\ ?$$

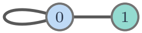
General source

When does a disjoint union of cliques minimize

$$\hom(G, H)^{1/v(G)}\ ?$$

Antiferromagnetic graphs

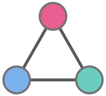
Antiferromagnetic: The adjacency matrix of H has at most one positive eigenvalue.



Independent sets

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

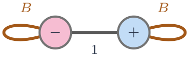
eigenvalues $\frac{1\pm\sqrt{5}}{2}$



Proper colorings

$$A(K_q) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

eigenvalues $q-1, -1, \ldots, -1$



Antiferromagnetic Ising

$$H_B = \begin{pmatrix} B & 1 \\ 1 & B \end{pmatrix}$$

$0 \le B \le 1$
eigenvalues $B+1, B-1$

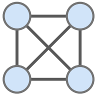
Main result

(Lee–S. 2026+) Every antiferromagnetic weighted graph H is **clique-minimizing**.

Regular case

$$\hom(G, H)^{1/v(G)} \geq \hom(K_{d+1}, H)^{1/(d+1)}.$$

The clique has the fewest configurations *per vertex*.



Main result

(Lee–S. 2026+) Every antiferromagnetic weighted graph H is **clique-minimizing**.

Regular case $\hom(G,H)^{1/v(G)} \geq \hom(K_{d+1},H)^{1/(d+1)}.$

Degree-by-degree form

$$\hom(G,H) \;\geq \prod_{v\in V(G)} \hom(K_{d_v+1},H)^{1/(d_v+1)}.$$

Each vertex contributes the clique determined by its own degree.

Stronger: the same comparison holds with vertex-dependent weights.

From separate phenomena to one criterion

2014
independent sets
regular graphs

2017
proper colorings
regular graphs

2019
independence polynomial
arbitrary degrees

2024
Ising models
partial results

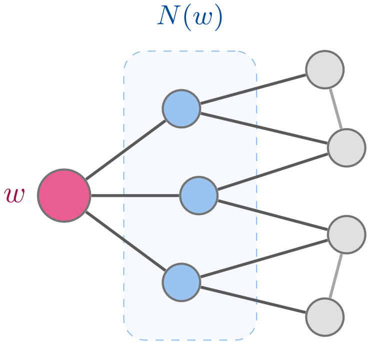
2026
semiproper colorings
special targets

2026
all antiferromagnetic targets

Sources: Cutler–Radcliffe; Csikvári; Sah–Sawhney–Stoner–Zhao; Davies–LeBlanc; Lee–Seo.

Proof ideas

Recurrence relation



$$i(G) = i\big(G - \{w\}\big) + i\big(G - (\{w\} \cup N(w))\big).$$

Lorentzian polynomials

- Introduced by **Brändén–Huh (2020)**.
- A framework for understanding **negative dependence**,
  which lies between **algebraic combinatorics** and **algebraic geometry**.

Our application:

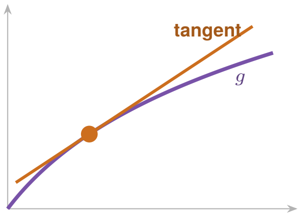
antiferromagnetic

↓

Lorentzian clique partition functions

↓

log-concavity



Entropy

Delete-one Shearer's lemma. For $X = (X_1, \ldots, X_m)$,

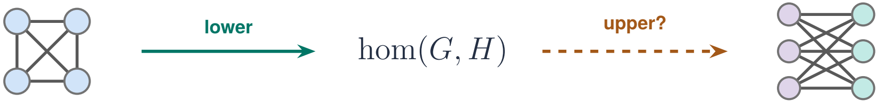
$$\sum_{j=1}^{m} \mathrm{H}\big(X_{[m]\setminus\{j\}}\big) \;\geq\; (m-1)\mathrm{H}(X).$$

For exchangeable Lorentzian measures, we prove a **strengthening** using **relative entropy contraction**:

$$\mathrm{D}(\mu_1 \| \nu_1) \;\leq\; \frac{1}{m}\mathrm{D}(\mu \| \nu).$$

A multiset analogue of *entropic independence* of Anari–Jain–Koehler–Pham–Vuong (2022).

The landscape and open questions



clique extremizers
proved lower bound

biclique extremizers
conjectured upper bound

1
Prove the biclique-maximizing conjecture for antiferromagnetic graphs.

2
Characterize all clique-minimizing graphs.

3
Determine the equality and near-equality cases.