Clique-minimizing models

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Joint with Joonkyung Lee

Flags in the Mountains

CSU Mountain Campus

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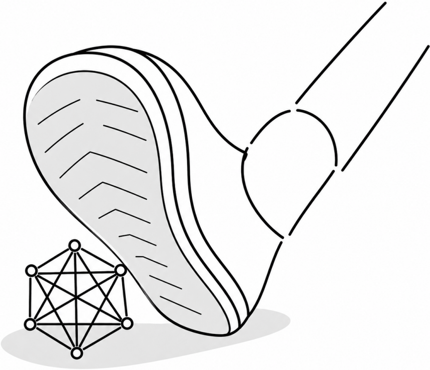

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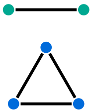
$\mathcal{G}$	Maximizer	Minimizer
d-regular bipartite	$K_{d,d}$ (Kahn '01)	infinite d-regular tree (Csikvári '16+)
d-regular	$K_{d,d}$ (Zhao '10)	K_{d+1} (Cutler–Radcliffe '14)
general graphs	bicliques (SSSZ '19)	cliques (SSSZ '19)

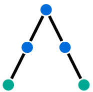
Number of independent sets: irregular graphs

Theorem (Sah–Sawhney–Stoner–Zhao '19)

Among the graphs with a **fixed degree distribution**, the disjoint union of cliques minimizes $i(G)^{1/v(G)}$.

$$\prod_{v \in V(G)} i(K_{d_v+1})^{1/(d_v+1)} \leq i(G)$$





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Among the graphs with a **fixed degree distribution**, the disjoint union of cliques minimizes $i(G)^{1/v(G)}$.

Among the graphs with a **fixed degree-degree distribution**, the disjoint union of bicliques maximizes $i(G)^{1/e(G)}$.

$$\prod_{v \in V(G)} i(K_{d_v+1})^{1/(d_v+1)} \le i(G) \le \prod_{uv \in E(G)} i(K_{d_u,d_v})^{1/(d_u d_v)}$$

Number of independent sets: irregular graphs

Theorem (Sah–Sawhney–Stoner–Zhao '19)

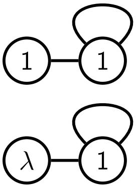
$$\prod_{v \in V(G)} \hom(K_{d_v+1}, \bullet\!\!-\!\!\overset{\circlearrowleft}{\bullet})^{1/(d_v+1)} \le i(G) \le \prod_{uv \in E(G)} \hom(K_{d_u,d_v}, \bullet\!\!-\!\!\overset{\circlearrowleft}{\bullet})^{1/(d_u d_v)}$$

$\bullet\!\!-\!\!\overset{\circlearrowleft}{\bullet}$ is **clique-minimizing** and **biclique-maximizing**.

Independence polynomials

$\mathcal{I}(G) := \{$the independent sets in $G\}$. $i(G) = \#\mathcal{I}(G) = \sum_{I \in \mathcal{I}(G)} 1.$

Independence polynomial $Z_G(\lambda) := \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}.$



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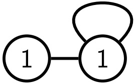

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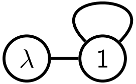
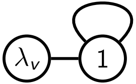


Multivariate independence polynomial



$$Z_G(\lambda_v : v \in V(G)) := \sum_{I \in \mathcal{I}(G)} \prod_{v \in I} \lambda_v.$$

Inequalities on independence polynomials

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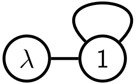

Theorem (Sah–Sawhney–Stoner–Zhao '19) For $\lambda \geq 0$,

$$\prod_{v \in V(G)} Z_{K_{d_v+1}}(\lambda)^{1/(d_v+1)} \leq Z_G(\lambda) \leq \prod_{uv \in E(G)} Z_{K_{d_u,d_v}}(\lambda)^{1/(d_u d_v)}.$$

Results

Theorem (Lee–S. '26+) For $\lambda_v \geq 0$, $v \in V(G)$,

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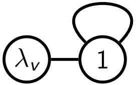

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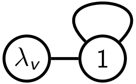

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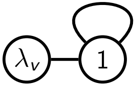

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Theorem (Lee–S. '26++) More graphs are clique-minimizing!

Summary

Q. Which graph G has the minimum (normalized) $i(G)$?

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Conjecture (Lee–S.) Every edge-weighted **antiferromagnetic** H is clique-minimizing:

$$\hom(G, H) \geq \prod_{v \in V(G)} \hom(K_{d_v+1}, H)^{1/(d_v+1)} \qquad \forall\ G$$

For the multivariate weighted setting, we proved this when H is:

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For the multivariate weighted setting, we proved this when H is:

1. (Lee–S. '26+) K_2° () and K_3° ();
2. (Lee–S. '26++) The antiferromagnetic Ising models () and K_q° with the loop weighted by $\theta \leq 1$.

Proof outline

Induction on $|V(G)|$ + Recurrence at max-degree vertex w, e.g.,

$$Z_G(\lambda) = Z_{G-w}(\lambda) + \lambda \cdot Z_{G-w-N(w)}(\lambda)$$

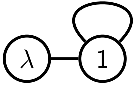
$\implies$ Reduce to a local inequality on w and $N(w)$.

1. K_2° () and K_3° (): More rely on the explicit polynomials.
2. Ising (), K_q° with the weighted loop: Properties of stable polynomials, strong-Rayleigh measures, Entropy independence, …….

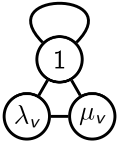
Appendix

Recurrence relations

- $Z_G(\lambda) = Z_{G-w}(\lambda) + \lambda \cdot Z_{G-w-N(w)}(\lambda)$



- $Z_G^{(2)}(\lambda_v, \mu_v : v) = Z_{G-w}^{(2)}(\lambda_v, \mu_v : v) + \lambda_w \cdot Z_{G-w}^{(2)}(\lambda_v, \mu_v : v)|_{\lambda_v = 0\ \forall v \in N(w)}$
$+ \mu_w \cdot Z_{G-w}^{(2)}(\lambda_v, \mu_v : v)|_{\mu_v = 0\ \forall v \in N(w)}$



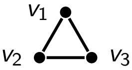
Other maximizers/minimizers

- (Cohen–Perkins–Tetali '17) $G =$  (Widom–Rowlinson model): K_{d+1} is the maximizer among the d-regular graphs.
- (Sernau '17) $\exists G$ s.t. neither $K_{d,d}$ nor K_{d+1} is a maximizer among the d-regular graphs.
- (Csikvári '16+)
  - $G =$ : the infinite d-regular tree is the minimizer among the d-regular graphs.
  - $G =$ : the infinite d-regular tree is the minimizer among the bipartite d-regular graphs.
- (Perarnau–Perkins '18)
  - $G =$ : the Petersen graph is the minimizer among the cubic graphs of girth ≥ 4.
  - $G =$ : the Heawood graph is the maximizer among the cubic graphs of girth ≥ 5.

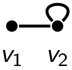
Antiferromagnetic graphs

A (edge-)weighted graph G is **antiferromagnetic** if its adjacency matrix has at most one positive eigenvalue.

$\iff$ $\langle \mathbf{x}, G\mathbf{x}\rangle\langle \mathbf{y}, G\mathbf{y}\rangle \leq \langle \mathbf{x}, G\mathbf{y}\rangle^2$ $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ with $\langle \mathbf{y}, G\mathbf{y}\rangle > 0$ (**hyperbolic**)



$$\rightsquigarrow \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$



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Conjecture (SSSZ '20) For all d-regular H and antiferromagnetic G,

$$\hom(H, G)^{1/v(H)} \leq \hom(K_{d,d}, G)^{1/(2d)}.$$

Why antiferromagnetic?

- (LO**S** '25+) Support of a weighted antiferromagnetic graph = blow-up of K_q or K_q°.
- (SSSZ '20) Every order 2 antiferromagnetic graph satisfies the conjecture.