

A lower bound on occupancy fractions

Jaehyeon Seo*

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Abstract

Davies, Sandhu, and Tan recently proved a lower bound on the expected size of an independent set under the hard-core model using the occupancy fraction method. Their bound is tight in terms of the degree sequence. We improve their fugacity range $[0, 3/(\Delta + 1)^2]$, where Δ is the maximum degree of the graph, to $[0, 0.0179/\Delta]$ by carefully analyzing the argument. After this note was written, Davies, Sandhu, the author, and Tan extended the bound to the multivariate setting with the fugacity range further improved to $[0, 1/\Delta)$, using a novel and cleaner method based on linear programming and matrix analysis.

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1 Introduction

Let G be a graph on the vertex set V , and let $\mathcal{I}(G)$ be the set of independent sets in G . The *occupancy fraction* is defined as

$$E_G(\lambda) := \frac{1}{n} \frac{\sum_{I \in \mathcal{I}(G)} |I| \lambda^{|I|}}{\sum_{I \in \mathcal{I}(G)} \lambda^{|I|}}.$$

Davies, Sandhu, and Tan [DST25] recently proved the following lower bound on the occupancy fraction.

Theorem 1 ([DST25, Theorem 1.3]). *Let G be a graph and let d_u be the degree of a vertex u in G . Then with $\Delta = \max_{u \in V(G)} d_u$, for any $\lambda \in [0, 3/(\Delta + 1)^2]$,*

$$\frac{1}{n} \sum_{u \in V(G)} \frac{\lambda}{1 + (d_u + 1)\lambda} = \frac{1}{n} \sum_{u \in V(G)} E_{K_{d_u+1}}(\lambda) \leq E_G(\lambda). \quad (1.1)$$

In this note, we improve the fugacity range $[0, 3/(\Delta + 1)^2]$ to $[0, c/\Delta]$ for a small constant $c > 0$, which can be chosen as 0.0179362. This is done by using exact truncation identities rather than the rough Taylor-expansion bounds used in [DST25], and by carefully estimating sums of cross-terms of the form $(\zeta_u - \zeta_v)(d_u - d_v)$ over the edges.

*Department of Mathematics, Yonsei University, Seoul, South Korea. Email: jaehyeonseo@yonsei.ac.kr.

After this note was written, Davies, Sandhu, the author, and Tan [DSST26] extended the inequality to the multivariate setting, with the fugacity range improved to $\lambda_u \in [0, 1/\Delta)$ for all $u \in V(G)$, using a novel and cleaner method based on linear programming and matrix analysis. Thus, the purpose of this note is to record how far the original approach of [DST25] can be pushed by a careful analysis.

Throughout the note, we assume $0 < \lambda \leq c/\Delta$, where $c \in (0, 1)$ will be chosen later.

1.1 Proof sketch

For $d \geq 0$, let

$$f(d) := \frac{\lambda}{1 + (d+1)\lambda}.$$

We follow [DST25] up to the point where Theorem 1 is reduced to proving the antisymmetrized edge-sum inequality

$$\mathcal{S}(\lambda) := \sum_{uv \in E} (d_u - d_v) T_{uv}(\lambda) \geq 0, \quad (1.2)$$

where

$$T_{uv}(\lambda) = \frac{f(d_v - 1)f(d_u)}{1 - \sum_{x \in N(u)} f(d_x - 1)} - \frac{f(d_u - 1)f(d_u)}{1 - \sum_{x \in N(v)} f(d_x - 1)}.$$

Note that [DST25] uses $\lambda \leq 3/(\Delta + 1)^2$ for this reduction, which can also be carried out under our assumption $\lambda \leq c/\Delta$ with $c \in (0, 1)$. We analyze this inequality under the weaker assumption $\lambda \leq c/\Delta$ for a sufficiently small absolute constant $c > 0$. Below, we outline the proof and highlight how our argument differs from that of [DST25].

(i) *Expand the numerators and denominators.* In [DST25], after rewriting the problem as a sum over edges, the denominators $1 - \sum_{x \in N(u)} f(d_x - 1)$ are bounded by simple degree-only expressions and the summand is then further lower-bounded by replacing neighbors' degrees with worst-case values. This produces a cleaner inequality but introduces losses of order Δ , leading to the range $\lambda = O(1/\Delta^2)$. In contrast, we do not collapse the neighborhood information into a worst-case bound. Instead, we expand both the numerator $A_u := f(d_u - 1)f(d_u)$ (Lemma 3) and the reciprocal denominator $B_u := (1 - F_u)^{-1}$ (with $F_u := \sum_{x \in N(u)} f(d_x - 1)$; Lemma 5) up to terms of order λ^4 and λ^2 , respectively, keeping the antisymmetry between (u, v) and (v, u) explicit.

(ii) *A structured leading term and the Laplacian norm $\|Ld\|_2$.* The expansion yields an explicit form (see (4.6)):

$$T_{uv}(\lambda) = 3(d_u - d_v)\lambda^3 + ((H_u - H_v) - (3(d_u + d_v) + 4)(d_u - d_v))\lambda^4 + R_{uv}(\lambda),$$

where $H_u = (Ld)_u = d_u^2 - \sum_{x \in N(u)} d_x$ and R_{uv} collects terms of order ≥ 5 . Here $L = \text{Diag}(d) - G$ is the combinatorial Laplacian of G . Multiplying by $(d_u - d_v)$ and summing over edges shows that the main contribution to $\mathcal{S}(\lambda)$ can be expressed in terms of the Dirichlet sum $\mathcal{E} := \sum_{uv \in E} (d_u - d_v)^2$ and the squared Laplacian norm $\|Ld\|_2^2$, via the identity

$$\sum_{uv \in E} (d_u - d_v)(H_u - H_v) = \|Ld\|_2^2.$$

(iii) *Split and bound the remainder terms.* A key technical difference is how one controls the error $R_{uv}(\lambda)$. Rather than bounding the full summand directly, we decompose the antisymmetrized remainder into a polynomial tail part and “structured” pieces involving the remainders from the expansions of A_u and B_u :

$$R_{uv} = R_{uv}^{\text{poly}} + R_{uv}^A + R_{uv}^B + R_{uv}^{AB}.$$

We then bound the global remainder $\sum_{uv \in E} (d_u - d_v)R_{uv}$ at the $\lambda^3 \mathcal{E}$ scale by combining: (a) Lipschitz-type estimates (Section 6.2) that turn differences like $|R_u(A) - R_v(A)|$ into $|d_u - d_v|$ -factors, and (b) energy estimates that control cross-terms like $(d_u - d_v)(H_u - H_v)$.

(iv) *Conclusion.* Combining the main term with the global remainder bounds shows that

$$\mathcal{S}(\lambda) \geq (1 - \phi(c))\lambda^3 \mathcal{E} \quad \text{whenever } \lambda \leq c/\Delta,$$

where $\phi(c) \rightarrow 0$ as $c \rightarrow 0$. This yields the desired improvement.

Notation

- $f(d) = \frac{\lambda}{1 + (d+1)\lambda}$
- $D_u = \sum_{x \in N(u)} d_x$
- $H_u = d_u^2 - D_u = \sum_{x \in N(u)} (d_u - d_x)$
 - $H = Ld$, where L is the combinatorial Laplacian and d is the degree function.
- $A_u = f(d_u - 1)f(d_u)$
- $A(d) = f(d-1)f(d) = \lambda^2 - (2d+1)\lambda^3 + (3d^2 + 3d + 1)\lambda^4 + R_d(A)$
- $F_u = \sum_{x \in N(u)} f(d_x - 1) = d_u\lambda - D_u\lambda^2 + R_u(F)$ (Lemma 4)
 - $R_u(F) = \lambda^3 \sum_{x \in N(u)} \frac{d_x^2}{1 + d_x\lambda}$
- $E_u = 1 - F_u$
- $B_u = 1/E_u = 1/(1 - F_u) = 1 + d_u\lambda + H_u\lambda^2 + R_u(B)$ (Lemma 5)
- $T_{uv}(\lambda) = \frac{A_v}{E_u} - \frac{A_u}{E_v} = \frac{f(d_v - 1)f(d_v)}{1 - \sum_{x \in N(u)} f(d_x - 1)} - \frac{f(d_u - 1)f(d_u)}{1 - \sum_{x \in N(v)} f(d_x - 1)}$
 - The goal (1.2) states $\sum_{uv \in E} (d_u - d_v)T_{uv}(\lambda) \geq 0$.
- $\mathcal{E} = \sum_{uv \in E} (d_u - d_v)^2$

2 Basic setup

For each $u \in V(G)$,

$$0 \leq F_u = \sum_{x \sim u} f(d_x - 1) = \sum_{x \in N(u)} \frac{\lambda}{1 + d_x\lambda} \leq \sum_{x \in N(u)} \lambda = d_u\lambda \leq c, \quad (2.1)$$

$$0 \leq D_u = \sum_{x \sim u} d_x \leq \Delta^2, \quad (2.2)$$

$$|H_u| \leq \sum_{x \in N(u)} |d_u - d_x| \leq \sum_{x \in N(u)} \Delta \leq \Delta^2. \quad (2.3)$$

We frequently use the following identities.

Lemma 2 (Exact truncation identities). *For every $t \geq 0$,*

$$\frac{1}{1+t} = 1 - t + \frac{t^2}{1+t}, \quad (2.4)$$

$$\frac{1}{1+t} = 1 - t + t^2 - \frac{t^3}{1+t}. \quad (2.5)$$

For every $z \in [0, 1)$,

$$\frac{1}{1-z} = 1 + z + z^2 + \frac{z^3}{1-z}. \quad (2.6)$$

In particular, if $0 \leq t \leq c$ then $|\frac{t^3}{1+t}| \leq t^3 \leq c^3$, and if $0 \leq z \leq c < 1$ then $0 \leq \frac{z^3}{1-z} \leq \frac{c^3}{1-c}$.

3 Expansions of basic terms

In this section, we estimate $A(d)$, F_u , and B_u using Lemma 2.

Lemma 3 (Expansion of $A(d)$). *Recall $A(d) = f(d-1)f(d)$. Define*

$$x_d := (2d+1)\lambda + d(d+1)\lambda^2 \geq 0 \quad \text{so that} \quad A(d) = \frac{\lambda^2}{1+x_d}.$$

Then

$$A(d) = \lambda^2 - (2d+1)\lambda^3 + (3d^2 + 3d + 1)\lambda^4 + R_d(A), \quad (3.1)$$

where

$$R_d(A) := \lambda^2 \left(2d(d+1)(2d+1)\lambda^3 + d^2(d+1)^2\lambda^4 - \frac{x_d^3}{1+x_d} \right) \quad (3.2)$$

and

$$|R_d(A)| \leq r_A(c)c^3\lambda^2 \quad \text{where} \quad r_A(c) \rightarrow 39 \text{ as } c \rightarrow 0.$$

Proof. This is a direct consequence of (2.5). □

Lemma 4 (Expansion of F_u). *Recall $F_u = \sum_{x \sim u} f(d_x - 1)$. Then*

$$F_u = d_u\lambda - D_u\lambda^2 + R_u(F) \quad (3.3)$$

where

$$R_u(F) := \lambda^3 \sum_{x \in N(u)} \frac{d_x^2}{1+d_x\lambda} \quad \text{and} \quad |R_u(F)| \leq c^3. \quad (3.4)$$

Proof. (2.4) gives

$$\frac{1}{1+d_x\lambda} = 1 - d_x\lambda + \frac{d_x^2\lambda^2}{1+d_x\lambda}.$$

Multiplying by λ and summing over $x \in N(u)$ gives (3.3). In addition,

$$|R_u(F)| \leq \lambda^3 \sum_{x \in N(u)} d_x^2 \leq \lambda^3 \Delta^3 \leq c^3. \quad \square$$

Lemma 5 (Expansion of B_u). *Recall $B_u = 1/E_u = 1/(1 - F_u)$. Then*

$$B_u = 1 + d_u\lambda + H_u\lambda^2 + R_u(B) \quad (3.5)$$

where $R_u(B) := B_u - (1 + d_u\lambda + H_u\lambda^2)$ and

$$|R_u(B)| \leq \frac{c^3}{1-c} + 3c^3 + 2c^4 =: \Phi_B(c). \quad (3.6)$$

Proof. By (2.6),

$$B_u = \frac{1}{1-F_u} = 1 + F_u + F_u^2 + \frac{F_u^3}{1-F_u}.$$

Using $H_u = d_u^2 - D_u$,

$$B_u - (1 + d_u\lambda + H_u\lambda^2) = (F_u - d_u\lambda + D_u\lambda^2) + (F_u^2 - d_u^2\lambda^2) + \frac{F_u^3}{1-F_u}. \quad (3.7)$$

By Lemma 4, the first bracket in (3.7) equals $R_u(F)$, and its absolute value is $\leq c^3$.

For the second bracket,

$$F_u^2 - d_u^2\lambda^2 = (F_u + d_u\lambda)(F_u - d_u\lambda) = (F_u + d_u\lambda)(-D_u\lambda^2 + R_u(F)).$$

Since $0 \leq F_u \leq c$ by (2.1), $D_u \leq \Delta^2$ by (2.2), and $|R_u(F)| \leq c^3$ by (3.4),

$$|F_u^2 - d_u^2\lambda^2| \leq (c+c)(c^2+c^3) = 2c^3 + 2c^4.$$

Also, $0 \leq F_u \leq c < 1$ gives

$$0 \leq \frac{F_u^3}{1-F_u} \leq \frac{c^3}{1-c}.$$

Combining these estimates in (3.7) yields (3.6). □

4 Estimate of $T_{uv}(\lambda)$

For each edge $uv \in E(G)$, define

$$T_{uv}(\lambda) := A_v B_u - A_u B_v = \frac{A_v}{E_u} - \frac{A_u}{E_v} = \frac{f(d_v - 1) f(d_v)}{1 - \sum_{x \in N(u)} f(d_x - 1)} - \frac{f(d_u - 1) f(d_u)}{1 - \sum_{x \in N(v)} f(d_x - 1)}.$$

Let $R_v(A) := R_{d_v}(A)$. By Lemmas 3 and 5,

$$A_v = \lambda^2 + a_v \lambda^3 + b_v \lambda^4 + R_v(A), \quad B_u = 1 + d_u \lambda + H_u \lambda^2 + R_u(B),$$

where

$$a_v := -(2d_v + 1), \quad b_v := 3d_v^2 + 3d_v + 1. \quad (4.1)$$

We have

$$(\lambda^2 + a_v \lambda^3 + b_v \lambda^4)(1 + d_u \lambda + H_u \lambda^2) = \lambda^2 + (a_v + d_u) \lambda^3 + (b_v + a_v d_u + H_u) \lambda^4 + (b_v d_u + a_v H_u) \lambda^5 + b_v H_u \lambda^6.$$

Define

$$\text{Poly}_{vu}(\lambda) := \lambda^2 + (a_v + d_u) \lambda^3 + (b_v + a_v d_u + H_u) \lambda^4, \quad \text{Tail}_{vu}^{\text{poly}}(\lambda) := (b_v d_u + a_v H_u) \lambda^5 + b_v H_u \lambda^6, \quad (4.2)$$

so that

$$(\lambda^2 + a_v \lambda^3 + b_v \lambda^4)(1 + d_u \lambda + H_u \lambda^2) = \text{Poly}_{vu}(\lambda) + \text{Tail}_{vu}^{\text{poly}}(\lambda).$$

Then

$$A_v B_u - \text{Poly}_{vu}(\lambda) = \text{Tail}_{vu}^{\text{poly}}(\lambda) + (\lambda^2 + a_v \lambda^3 + b_v \lambda^4) R_u(B) + (1 + d_u \lambda + H_u \lambda^2) R_v(A) + R_v(A) R_u(B). \quad (4.3)$$

Define $R_{uv}(\lambda)$ so that

$$T_{uv}(\lambda) = \text{Poly}_{vu}(\lambda) - \text{Poly}_{uv}(\lambda) + R_{uv}(\lambda). \quad (4.4)$$

We claim

$$\text{Poly}_{vu}(\lambda) - \text{Poly}_{uv}(\lambda) = 3(d_u - d_v) \lambda^3 + ((H_u - H_v) - (3(d_u + d_v) + 4)(d_u - d_v)) \lambda^4. \quad (4.5)$$

By definition,

$$\text{Poly}_{vu}(\lambda) - \text{Poly}_{uv}(\lambda) = ((a_v + d_u) - (a_u + d_v)) \lambda^3 + ((b_v + a_v d_u + H_u) - (b_u + a_u d_v + H_v)) \lambda^4.$$

By (4.1), the λ^3 coefficient is

$$(a_v + d_u) - (a_u + d_v) = (-(2d_v + 1) + d_u) - (-(2d_u + 1) + d_v) = 3(d_u - d_v).$$

Also, the λ^4 coefficient is

$$\begin{aligned} & (b_v - b_u) + (a_v d_u - a_u d_v) + (H_u - H_v) \\ &= (3(d_v^2 - d_u^2) + 3(d_v - d_u)) + (-(2d_v + 1)d_u + (2d_u + 1)d_v) + (H_u - H_v) \\ &= -3(d_u + d_v)(d_u - d_v) - 3(d_u - d_v) - (d_u - d_v) + (H_u - H_v) \\ &= (H_u - H_v) - (3(d_u + d_v) + 4)(d_u - d_v). \end{aligned}$$

Thus, (4.5) follows.

Combining (4.4) and (4.5),

$$T_{uv}(\lambda) = 3(d_u - d_v) \lambda^3 + ((H_u - H_v) - (3(d_u + d_v) + 4)(d_u - d_v)) \lambda^4 + R_{uv}(\lambda). \quad (4.6)$$

5 Summing over edges

Let $L := \text{Diag}(d) - G$ be the combinatorial Laplacian, where $\text{Diag}(d)$ is the diagonal degree matrix and G is the adjacency matrix. Then $H = Ld$, i.e., $H_u = (Ld)_u$. Define the Dirichlet sum of the degree function:

$$\mathcal{E} := \sum_{uv \in E} (d_u - d_v)^2.$$

Recall that our goal (1.2) is to show

$$\mathcal{S}(\lambda) := \sum_{uv \in E} (d_u - d_v) T_{uv}(\lambda) \geq 0.$$

Equation (4.6) states that

$$T_{uv}(\lambda) = 3(d_u - d_v)\lambda^3 + ((H_u - H_v) - (3(d_u + d_v) + 4)(d_u - d_v))\lambda^4 + R_{uv}(\lambda).$$

Multiplying this by $(d_u - d_v)$ and summing over $uv \in E$ yields

$$\mathcal{S}(\lambda) = 3\lambda^3 \mathcal{E} + \lambda^4 \left(\sum_{uv \in E} (d_u - d_v)(H_u - H_v) - W \right) + R_{\geq 5}(\lambda), \quad (5.1)$$

where we define

$$W := \sum_{uv \in E} (3(d_u + d_v) + 4)(d_u - d_v)^2 \quad (5.2)$$

and

$$R_{\geq 5}(\lambda) := \sum_{uv \in E} (d_u - d_v) R_{uv}(\lambda). \quad (5.3)$$

Lemma 6. For any functions $a, b : V \rightarrow \mathbb{R}$ (viewed as column vectors in \mathbb{R}^V),

$$\sum_{uv \in E} (a_u - a_v)(b_u - b_v) = a^\top Lb.$$

Proof. The left-hand side is

$$\sum_{uv \in E} (a_u - a_v)(b_u - b_v) = \sum_{uv \in E} (a_u b_u + a_v b_v - a_u b_v - a_v b_u).$$

Since

$$\sum_{uv \in E} (a_u b_u + a_v b_v) = \sum_{u \in V} d_u a_u b_u \quad \text{and} \quad \sum_{uv \in E} (a_u b_v + a_v b_u) = \sum_{u \in V} \sum_{v \in N(u)} a_u b_v,$$

we have

$$\sum_{uv \in E} (a_u - a_v)(b_u - b_v) = \sum_{u \in V} d_u a_u b_u - \sum_{u \in V} \sum_{v \in N(u)} a_u b_v = a^\top (\text{Diag}(d) - G)b = a^\top Lb,$$

as claimed. \square

Apply Lemma 6 with $a = d$ (the degree function) and $b = H = Ld$ to get

$$\sum_{uv \in E} (d_u - d_v)(H_u - H_v) = d^\top LH = d^\top L(Ld) = d^\top L^2 d = \|Ld\|_2^2.$$

Thus, (5.1) becomes

$$\boxed{\mathcal{S}(\lambda) = 3\lambda^3 \mathcal{E} + \lambda^4 (\|Ld\|_2^2 - W) + R_{\geq 5}(\lambda)}. \quad (5.4)$$

Lemma 7. $0 \leq W \leq (6\Delta + 4) \mathcal{E}$.

Proof. Nonnegativity is immediate from the definition (5.2). For the upper bound,

$$W = \sum_{uv \in E} (3(d_u + d_v) + 4)(d_u - d_v)^2 \leq \sum_{uv \in E} (6\Delta + 4)(d_u - d_v)^2 = (6\Delta + 4)\mathcal{E}. \quad \square$$

6 Estimate of $R_{\geq 5}(\lambda)$

Recall from (5.3) that

$$R_{\geq 5}(\lambda) = \sum_{uv \in E} (d_u - d_v) R_{uv}(\lambda),$$

where, by (4.4),

$$R_{uv}(\lambda) = T_{uv}(\lambda) - \text{Poly}_{vu}(\lambda) + \text{Poly}_{uv}(\lambda) = (A_v B_u - \text{Poly}_{vu}(\lambda)) - (A_u B_v - \text{Poly}_{uv}(\lambda)).$$

6.1 Estimate of the polynomial tail

Recall from (4.2) that the tail of the polynomial part of $A_v B_u$ is defined as

$$\text{Tail}_{vu}^{\text{poly}}(\lambda) = (b_v d_u + a_v H_u) \lambda^5 + b_v H_u \lambda^6,$$

where $a_w = -(2d_w + 1)$ and $b_w = 3d_w^2 + 3d_w + 1$. Define

$$R_{uv}^{\text{poly}}(\lambda) := \text{Tail}_{vu}^{\text{poly}}(\lambda) - \text{Tail}_{uv}^{\text{poly}}(\lambda). \quad (6.1)$$

Also define the corresponding contribution to the global remainder:

$$R_{\geq 5}^{\text{poly}}(\lambda) := \sum_{uv \in E} (d_u - d_v) R_{uv}^{\text{poly}}(\lambda). \quad (6.2)$$

We first record some preparatory estimates. The combinatorial Laplacian L is symmetric positive semidefinite and satisfies $\|L\|_2 \leq 2\Delta$ (e.g., by Gershgorin's theorem). In particular, for any vector x ,

$$x^\top L^2 x \leq \|L\|_2 x^\top L x \leq 2\Delta x^\top L x, \quad (6.3)$$

and likewise

$$x^\top L^3 x \leq \|L\|_2^2 x^\top L x \leq (2\Delta)^2 x^\top L x. \quad (6.4)$$

Lemma 6 with $a = b = x$ gives

$$\sum_{uv \in E} (x_u - x_v)^2 = x^\top L x.$$

Lemma 8.

$$\sum_{uv \in E} |(d_u - d_v)(H_u - H_v)| \leq 2\Delta \mathcal{E}.$$

Proof. The Cauchy–Schwarz inequality gives

$$\sum_{uv \in E} |(d_u - d_v)(H_u - H_v)| \leq \left(\sum_{uv \in E} (d_u - d_v)^2 \right)^{1/2} \left(\sum_{uv \in E} (H_u - H_v)^2 \right)^{1/2}.$$

The first factor is $\mathcal{E}^{1/2}$. For the second factor, apply Lemma 6 with $a = b = H$ to get $\sum_{uv \in E} (H_u - H_v)^2 = H^\top L H$. Since $H = Ld$, we have $H^\top L H = d^\top L^3 d$, and (6.4) gives $d^\top L^3 d \leq (2\Delta)^2 d^\top L d = (2\Delta)^2 \mathcal{E}$. Thus, the claim follows. \square

We now bound $|R_{\geq 5}^{\text{poly}}(\lambda)|$.

Lemma 9 (The polynomial tail contributes at the $\lambda^3 \mathcal{E}$ scale).

$$|R_{\geq 5}^{\text{poly}}(\lambda)| \leq (11c^2 + 23c^3) \lambda^3 \mathcal{E}.$$

Proof. By (6.1),

$$R_{uv}^{\text{poly}}(\lambda) = ((b_v d_u - b_u d_v) + (a_v H_u - a_u H_v)) \lambda^5 + (b_v H_u - b_u H_v) \lambda^6.$$

Step 1: Bound the λ^5 term. Putting $b_w = 3d_w^2 + 3d_w + 1$,

$$b_v d_u - b_u d_v = (d_u - d_v)(1 - 3d_u d_v).$$

Also, since $a_w = -(2d_w + 1)$,

$$\begin{aligned} a_v H_u - a_u H_v &= -(2d_v + 1)H_u + (2d_u + 1)H_v \\ &= (d_u - d_v)(H_u + H_v) + (d_u + d_v + 1)(H_v - H_u). \end{aligned}$$

Therefore,

$$\begin{aligned} (d_u - d_v)((b_v d_u - b_u d_v) + (a_v H_u - a_u H_v)) \lambda^5 &= \lambda^5 \left[(d_u - d_v)^2 (1 - 3d_u d_v) \right. \\ &\quad \left. + (d_u - d_v)^2 (H_u + H_v) \right. \\ &\quad \left. + (d_u - d_v)(d_u + d_v + 1)(H_v - H_u) \right]. \end{aligned} \quad (6.5)$$

Summing (6.5) over $uv \in E$, we bound the three terms as follows.

- $\Delta \geq 1$ gives $|1 - 3d_u d_v| \leq 3\Delta^2$, so

$$\sum_{uv \in E} (d_u - d_v)^2 |1 - 3d_u d_v| \leq 3\Delta^2 \mathcal{E}.$$

- By (2.3), $|H_u| \leq \Delta^2$, hence $|H_u + H_v| \leq 2\Delta^2$, so

$$\sum_{uv \in E} (d_u - d_v)^2 |H_u + H_v| \leq 2\Delta^2 \mathcal{E}.$$

- Finally, $d_u + d_v + 1 \leq 2\Delta + 1 \leq 3\Delta$, so by Lemma 8,

$$\sum_{uv \in E} |(d_u - d_v)(d_u + d_v + 1)(H_v - H_u)| \leq 3\Delta \sum_{uv \in E} |(d_u - d_v)(H_u - H_v)| \leq 6\Delta^2 \mathcal{E}.$$

Multiplying by λ^5 shows that the total λ^5 contribution to $|R_{\geq 5}^{\text{poly}}(\lambda)|$ is at most

$$(3 + 2 + 6)\Delta^2 \lambda^5 \mathcal{E} = 11\Delta^2 \lambda^5 \mathcal{E} \leq 11c^2 \lambda^3 \mathcal{E}.$$

Step 2: Bound the λ^6 term. We write

$$b_v H_u - b_u H_v = (b_v - b_u)H_u + b_u(H_u - H_v).$$

For the first part, since $b_v - b_u = (d_v - d_u)(3(d_u + d_v) + 3)$ and $|H_u| \leq \Delta^2$ by (2.3),

$$\begin{aligned} \sum_{uv \in E} |(d_u - d_v)(b_v - b_u)H_u| &= \sum_{uv \in E} (d_u - d_v)^2 (3(d_u + d_v) + 3) |H_u| \\ &\leq \sum_{uv \in E} (d_u - d_v)^2 \cdot 9\Delta \cdot \Delta^2 \\ &= 9\Delta^3 \mathcal{E}. \end{aligned}$$

For the second part, using $b_u = 3d_u^2 + 2d_u + 1 \leq 7\Delta^2$ together with Lemma 8,

$$\begin{aligned} \sum_{uv \in E} |(d_u - d_v)b_u(H_u - H_v)| &\leq \sum_{uv \in E} |(d_u - d_v)(H_u - H_v)| \cdot 7\Delta^2 \\ &\leq 14\Delta^3 \mathcal{E}. \end{aligned}$$

Thus, the λ^6 contribution to $|R_{\geq 5}^{\text{poly}}(\lambda)|$ is at most

$$(9 + 14)\Delta^3 \lambda^6 \mathcal{E} \leq 23c^3 \lambda^3 \mathcal{E}.$$

Combining Steps 1 and 2 proves the lemma. □

6.2 Lipschitz and Lipschitz-type bounds for R_A and R_B

Lemma 9 shows that the polynomial-tail terms of orders λ^5 and λ^6 already satisfy a bound at the $\lambda^3 \mathcal{E}$ scale after summing. To obtain an analogous bound at the $\lambda^3 \mathcal{E}$ scale for the full $R_{\geq 5}(\lambda)$, we decompose $R_{uv} = R_{uv}^{\text{poly}} + R_{uv}^{\text{rest}}$, expand R_{uv}^{rest} into antisymmetrized pieces involving $R_u(A)$ and $R_u(B)$, and then bound $\sum_{uv \in E} (d_u - d_v) R_{uv}^{\text{rest}}$ using:

- Lipschitz bounds (Lemmas 10 and 11) to produce factors of $|d_u - d_v|$, and
- energy estimates for cross-terms of the form $(d_u - d_v)(H_u - H_v)$, as in Lemma 8.

Lemma 10 (Lipschitz bound for R_A). *Let $0 \leq d, e \leq \Delta$. Then*

$$|R_d(A) - R_e(A)| \leq C_A(c) \lambda^3 |d - e|,$$

where $C_A(c) := 1663c^2$.

Proof. Recall

$$R_d(A) = \lambda^2 \left(2d(d+1)(2d+1)\lambda^3 + d^2(d+1)^2\lambda^4 - \frac{x_d^3}{1+x_d} \right)$$

as defined in (3.2), where $x_d = (2d+1)\lambda + d(d+1)\lambda^2$. Then

$$|R_d(A) - R_e(A)| \leq |2d(d+1)(2d+1) - 2e(e+1)(2e+1)|\lambda^5 + |d^2(d+1)^2 - e^2(e+1)^2|\lambda^6 + \left| \frac{x_d^3}{1+x_d} - \frac{x_e^3}{1+x_e} \right| \lambda^2.$$

We bound each term from above separately.

First, $2d(d+1)(2d+1) = 4d^3 + 6d^2 + 2d$, so

$$\begin{aligned} |2d(d+1)(2d+1) - 2e(e+1)(2e+1)| &\leq 4|d^3 - e^3| + 6|d^2 - e^2| + 2|d - e| \leq (4 \cdot 3\Delta^2 + 6 \cdot 2\Delta + 2)|d - e| \\ &\leq 26\Delta^2|d - e|. \end{aligned}$$

Second,

$$\begin{aligned} |d^2(d+1)^2 - e^2(e+1)^2| &= (d(d+1) + e(e+1))|d(d+1) - e(e+1)| = (d(d+1) + e(e+1))(d+e+1)|d - e| \\ &\leq 4\Delta^2 \cdot 3\Delta \cdot |d - e| = 12\Delta^3|d - e|. \end{aligned}$$

Third, write $\psi(x) := \frac{x^3}{1+x}$ for $x \geq 0$. A direct calculation gives

$$\psi'(x) = \frac{x^2(3+2x)}{(1+x)^2} \leq x^2(3+2x).$$

Also, $x_d \leq 3c + 2c^2 < 5c$, and

$$\sup_{x \in [0, 5c]} \psi'(x) \leq (5c)^2(3 + 2 \cdot 5c) \leq 325c^2.$$

Moreover,

$$|x_d - x_e| \leq |(2d+1) - (2e+1)|\lambda + |d(d+1) - e(e+1)|\lambda^2 \leq 2|d - e|\lambda + 3\Delta|d - e|\lambda^2 \leq 5|d - e|,$$

so the mean value theorem yields

$$|\psi(x_d) - \psi(x_e)| \leq \sup_{x \in [0, 5c]} \psi'(x) |x_d - x_e| \leq 1625c^2\lambda |d - e|.$$

Combining the estimates, we have

$$\begin{aligned} |R_d(A) - R_e(A)| &\leq 26\Delta^2|d - e| \cdot \lambda^5 + 12\Delta^3|d - e| \cdot \lambda^6 + 1625c^2\lambda |d - e| \cdot \lambda^2 \leq (26c^2 + 12c^3 + 1625c^2)\lambda^3 |d - e| \\ &\leq 1663c^2\lambda^3 |d - e|. \end{aligned} \quad \square$$

Lemma 11 (Lipschitz-type bound for R_B). *For any vertices $u, v \in V$,*

$$|R_u(B) - R_v(B)| \leq C_{B,1}(c) \lambda |d_u - d_v| + C_{B,2}(c) \lambda^2 |H_u - H_v| + C_{B,3}(c) |R_u(F) - R_v(F)|,$$

where $R_u(F)$ is as in Lemma 4 and the coefficients are

$$C_{B,1}(c) := 1 + \frac{1+2c}{(1-c)^2}, \quad C_{B,2}(c) := 1 + \frac{1}{(1-c)^2}, \quad C_{B,3}(c) := \frac{1}{(1-c)^2}.$$

Proof. Recall $B_u = (1 - F_u)^{-1}$ and $R_u(B) = B_u - (1 + d_u\lambda + H_u\lambda^2)$ from (3.5). Therefore

$$|R_u(B) - R_v(B)| \leq |B_u - B_v| + \lambda |d_u - d_v| + \lambda^2 |H_u - H_v|. \quad (6.6)$$

Next,

$$B_u - B_v = \frac{1}{1 - F_u} - \frac{1}{1 - F_v} = \frac{F_u - F_v}{(1 - F_u)(1 - F_v)}.$$

Since $0 \leq F_u, F_v \leq c$, we have $(1 - F_u)(1 - F_v) \geq (1 - c)^2$, so

$$|B_u - B_v| \leq \frac{|F_u - F_v|}{(1 - c)^2}. \quad (6.7)$$

Finally, by Lemma 4 we may write

$$F_u = d_u\lambda - D_u\lambda^2 + R_u(F), \quad F_v = d_v\lambda - D_v\lambda^2 + R_v(F),$$

so

$$F_u - F_v = \lambda(d_u - d_v) - \lambda^2(D_u - D_v) + (R_u(F) - R_v(F)).$$

Using $D_u = d_u^2 - H_u$ and $D_v = d_v^2 - H_v$, we have

$$D_u - D_v = (d_u^2 - d_v^2) - (H_u - H_v) = (d_u - d_v)(d_u + d_v) - (H_u - H_v),$$

hence

$$F_u - F_v = \lambda(d_u - d_v) - \lambda^2(d_u - d_v)(d_u + d_v) + \lambda^2(H_u - H_v) + (R_u(F) - R_v(F)).$$

Therefore, using $|d_u + d_v| \leq 2\Delta$ and $\lambda\Delta \leq c$,

$$|F_u - F_v| \leq \lambda(1 + 2c)|d_u - d_v| + \lambda^2|H_u - H_v| + |R_u(F) - R_v(F)|. \quad (6.8)$$

Combining (6.6), (6.7), and (6.8) yields

$$\begin{aligned} |R_u(B) - R_v(B)| &\leq \frac{1}{(1-c)^2} \left(\lambda(1+2c)|d_u - d_v| + \lambda^2|H_u - H_v| + |R_u(F) - R_v(F)| \right) + \lambda|d_u - d_v| + \lambda^2|H_u - H_v| \\ &= \left(1 + \frac{1+2c}{(1-c)^2} \right) \lambda|d_u - d_v| + \left(1 + \frac{1}{(1-c)^2} \right) \lambda^2|H_u - H_v| + \frac{1}{(1-c)^2} |R_u(F) - R_v(F)|. \quad \square \end{aligned}$$

Remark. Unlike $R_u(A)$, the term $R_u(B)$ depends on the third-order neighborhood quantity $R_u(F)$ from Lemma 4; consequently, a bound purely in terms of $|d_u - d_v|$ and $|H_u - H_v|$ is not available without additional control on $|R_u(F) - R_v(F)|$. This is done in Lemma 12.

6.3 A structured global bound for $R_{\geq 5}(\lambda)$

The goal of this subsection is to show

$$\boxed{|R_{\geq 5}(\lambda)| \leq \Psi(c) \lambda^3 \mathcal{E} + \Xi(c) \lambda^2 \mathcal{H}(\lambda);}$$

the terms $\Psi(c)$, $\Xi(c)$, and $\mathcal{H}(\lambda)$ will be defined below.

Let

$$\tilde{R}_{vu}(\lambda) := A_v B_u - \text{Poly}_{vu}(\lambda)$$

so that, by definition (4.4) of $R_{uv}(\lambda)$,

$$\begin{aligned} R_{uv}(\lambda) &= T_{uv}(\lambda) - \text{Poly}_{vu}(\lambda) + \text{Poly}_{uv}(\lambda) \\ &= (A_v B_u - A_u B_v) - \text{Poly}_{vu}(\lambda) + \text{Poly}_{uv}(\lambda) \\ &= \tilde{R}_{vu}(\lambda) - \tilde{R}_{uv}(\lambda). \end{aligned}$$

Recall from Lemmas 3 and 5 that

$$A_v = \lambda^2 + a_v \lambda^3 + b_v \lambda^4 + R_v(A), \quad B_u = 1 + d_u \lambda + H_u \lambda^2 + R_u(B),$$

where $a_v = -(2d_v + 1)$ and $b_v = 3d_v^2 + 3d_v + 1$. Let

$$A_v^{\leq 4} := \lambda^2 + a_v \lambda^3 + b_v \lambda^4, \quad B_u^{\leq 2} := 1 + d_u \lambda + H_u \lambda^2, \quad (6.9)$$

so that $A_v = A_v^{\leq 4} + R_v(A)$ and $B_u = B_u^{\leq 2} + R_u(B)$. Also, by (4.2),

$$\text{Poly}_{vu}(\lambda) = \lambda^2 + (a_v + d_u) \lambda^3 + (b_v + a_v d_u + H_u) \lambda^4 = A_v^{\leq 4} B_u^{\leq 2} - \text{Tail}_{vu}^{\text{poly}}(\lambda).$$

Using $|a_v| \leq 3\Delta$, $b_v \leq 7\Delta^2$, $|H_u| \leq \Delta^2$, and $\lambda\Delta \leq c$, we record the bounds

$$|A_v^{\leq 4}| = |\lambda^2 + a_v \lambda^3 + b_v \lambda^4| \leq \lambda^2(1 + 3c + 7c^2) = \lambda^2 M_A(c), \quad (6.10)$$

$$|B_u^{\leq 2}| = |1 + d_u \lambda + H_u \lambda^2| \leq 1 + c + c^2 = M_B(c), \quad (6.11)$$

where

$$M_A(c) := 1 + 3c + 7c^2, \quad M_B(c) := 1 + c + c^2. \quad (6.12)$$

Decomposition of the antisymmetrized remainder

Expanding $A_v B_u$ using (6.9) and subtracting Poly_{vu} gives

$$\tilde{R}_{vu} = \underbrace{\text{Tail}_{vu}^{\text{poly}}}_{\text{pure polynomial tail}} + \underbrace{A_v^{\leq 4} R_u(B)}_{R_B \text{ part}} + \underbrace{B_u^{\leq 2} R_v(A)}_{R_A \text{ part}} + \underbrace{R_v(A) R_u(B)}_{R_A \cdot R_B \text{ part}}.$$

Therefore,

$$R_{uv} = \tilde{R}_{vu}(\lambda) - \tilde{R}_{uv}(\lambda) = R_{uv}^{\text{poly}} + R_{uv}^A + R_{uv}^B + R_{uv}^{AB},$$

where

$$\begin{aligned} R_{uv}^{\text{poly}} &:= \text{Tail}_{vu}^{\text{poly}} - \text{Tail}_{uv}^{\text{poly}} \quad (\text{as defined in (6.1)}), \\ R_{uv}^A &:= B_u^{\leq 2} R_v(A) - B_v^{\leq 2} R_u(A), \\ R_{uv}^B &:= A_v^{\leq 4} R_u(B) - A_u^{\leq 4} R_v(B), \\ R_{uv}^{AB} &:= R_v(A) R_u(B) - R_u(A) R_v(B). \end{aligned}$$

Consequently,

$$|R_{\geq 5}(\lambda)| \leq |R_{\geq 5}^{\text{poly}}(\lambda)| + |R_{\geq 5}^A(\lambda)| + |R_{\geq 5}^B(\lambda)| + |R_{\geq 5}^{AB}(\lambda)|, \quad (6.13)$$

where $R_{\geq 5}^*(\lambda) := \sum_{uv \in E} (d_u - d_v) R_{uv}^*(\lambda)$ for $* \in \{\text{poly}, A, B, AB\}$. Note that $R_{\geq 5}^{\text{poly}}(\lambda)$ was already defined in (6.2). We bound each term on the right-hand side.

(1) *Polynomial tail.* By Lemma 9,

$$|R_{\geq 5}^{\text{poly}}(\lambda)| \leq (11c^2 + 23c^3) \lambda^3 \mathcal{E}. \quad (6.14)$$

(2) *The R^A part.* Write

$$R_{uv}^A = (B_u^{\leq 2} - B_v^{\leq 2}) R_v(A) + B_v^{\leq 2} (R_v(A) - R_u(A)).$$

For the first term, since $B_u^{\leq 2} - B_v^{\leq 2} = \lambda(d_u - d_v) + \lambda^2(H_u - H_v)$,

$$\begin{aligned} \sum_{uv \in E} |d_u - d_v| |(B_u^{\leq 2} - B_v^{\leq 2}) R_v(A)| &\leq \sum_{uv \in E} |d_u - d_v| |R_v(A)| (\lambda |d_u - d_v| + \lambda^2 |H_u - H_v|) \\ &= \sum_{uv \in E} \lambda |R_v(A)| |d_u - d_v|^2 + \sum_{uv \in E} \lambda^2 |R_v(A)| |d_u - d_v| |H_u - H_v| \\ &\leq \lambda^3 r_A(c) c^3 \mathcal{E} + \lambda^4 r_A(c) c^3 \cdot 2\Delta \mathcal{E} \\ &\leq r_A(c) (1 + 2c) c^3 \lambda^3 \mathcal{E}, \end{aligned}$$

using $|R_A(d)| \leq r_A(c) c^3 \lambda^2$ (Lemma 3) and Lemma 8.

For the second term, Lemma 10 and $|B_v^{\leq 2}| \leq M_B(c)$ from (6.11) give

$$\begin{aligned} \sum_{uv \in E} |d_u - d_v| |B_v^{\leq 2} (R_v(A) - R_u(A))| &\leq \sum_{uv \in E} |d_u - d_v| \cdot M_B(c) \cdot C_A(c) \lambda^3 |d_u - d_v| \\ &= M_B(c) C_A(c) \lambda^3 \mathcal{E}. \end{aligned}$$

Therefore,

$$|R_{\geq 5}^A(\lambda)| = \left| \sum_{uv \in E} (d_u - d_v) R_{uv}^A(\lambda) \right| \leq (M_B(c) C_A(c) + r_A(c) (1 + 2c) c^3) \lambda^3 \mathcal{E}. \quad (6.15)$$

(3) *The R^B part.* Write

$$R_{uv}^B = (A_v^{\leq 4} - A_u^{\leq 4}) R_u(B) + A_u^{\leq 4} (R_u(B) - R_v(B)).$$

For the first term, using the definition $A_w^{\leq 4} = \lambda^2 + a_w \lambda^3 + b_w \lambda^4$ in (6.9) together with $|a_v - a_u| = 2|d_u - d_v|$ and $|b_v - b_u| = |(d_v - d_u)(3(d_u + d_v) + 3)| \leq 9\Delta |d_u - d_v|$,

$$|A_v^{\leq 4} - A_u^{\leq 4}| = |(a_v - a_u) \lambda^3 + (b_v - b_u) \lambda^4| \leq 2\lambda^3 |d_u - d_v| + 9\Delta \lambda^4 |d_u - d_v| \leq (2 + 9c) \lambda^3 |d_u - d_v|.$$

With $|R_u(B)| \leq \Phi_B(c)$ by (3.6), this gives

$$\sum_{uv \in E} |d_u - d_v| |(A_v^{\leq 4} - A_u^{\leq 4}) R_u(B)| \leq (2 + 9c) \Phi_B(c) \lambda^3 \mathcal{E}. \quad (6.16)$$

For the second term, by Lemma 11,

$$|R_u(B) - R_v(B)| \leq C_{B,1}(c) \lambda |d_u - d_v| + C_{B,2}(c) \lambda^2 |H_u - H_v| + C_{B,3}(c) |R_u(F) - R_v(F)|.$$

Hence, using $|A_u^{\leq 4}| \leq \lambda^2 M_A(c)$ by (6.10),

$$\begin{aligned} \sum_{uv \in E} |d_u - d_v| |A_u^{\leq 4}(R_u(B) - R_v(B))| &\leq M_A(c) C_{B,1}(c) \lambda^3 \mathcal{E} \\ &+ M_A(c) C_{B,2}(c) \lambda^4 \sum_{uv \in E} |d_u - d_v| |H_u - H_v| \\ &+ M_A(c) C_{B,3}(c) \lambda^2 \sum_{uv \in E} |d_u - d_v| |R_u(F) - R_v(F)|. \end{aligned} \quad (6.17)$$

Using Lemma 8 in the middle line gives

$$\begin{aligned} M_A(c) C_{B,2}(c) \lambda^4 \sum_{uv \in E} |d_u - d_v| |H_u - H_v| &\leq M_A(c) C_{B,2}(c) \lambda^4 \cdot 2\Delta \mathcal{E} \\ &\leq 2c M_A(c) C_{B,2}(c) \lambda^3 \mathcal{E}. \end{aligned} \quad (6.18)$$

Combining (6.16), (6.17), and (6.18),

$$\begin{aligned} |R_{\geq 5}^B(\lambda)| &= \left| \sum_{uv \in E} (d_u - d_v) R_{uv}^B(\lambda) \right| \\ &\leq \left((2 + 9c) \Phi_B(c) + M_A(c) (C_{B,1}(c) + 2c C_{B,2}(c)) \right) \lambda^3 \mathcal{E} + M_A(c) C_{B,3}(c) \lambda^2 \mathcal{H}(\lambda), \end{aligned} \quad (6.19)$$

where

$$\mathcal{H}(\lambda) := \sum_{uv \in E} |d_u - d_v| |R_u(F) - R_v(F)|.$$

(4) *The $R^A \cdot R^B$ part.* Similarly,

$$R_{uv}^{AB} = (R_v(A) - R_u(A)) R_u(B) + R_u(A) (R_u(B) - R_v(B)).$$

Using Lemmas 8, 10, and 11, $|R_u(B)| \leq \Phi_B(c)$ by (3.6), and $|R_u(A)| \leq r_A(c) c^3 \lambda^2$ by Lemma 3,

$$\begin{aligned} |R_{\geq 5}^{AB}(\lambda)| &\leq \sum_{uv \in E} |d_u - d_v| \cdot |R_v(A) - R_u(A)| |R_u(B)| + \sum_{uv \in E} |d_u - d_v| \cdot |R_u(A)| |R_u(B) - R_v(B)| \\ &\leq \sum_{uv \in E} |d_u - d_v| \cdot C_A(c) \lambda^3 |d_u - d_v| \cdot \Phi_B(c) \\ &\quad + \sum_{uv \in E} |d_u - d_v| \cdot r_A(c) c^3 \lambda^2 \cdot (C_{B,1}(c) \lambda |d_u - d_v| + C_{B,2}(c) \lambda^2 |H_u - H_v| + C_{B,3}(c) |R_u(F) - R_v(F)|) \\ &= C_A(c) \Phi_B(c) \lambda^3 \mathcal{E} + r_A(c) c^3 C_{B,1}(c) \lambda^3 \mathcal{E} + r_A(c) c^3 C_{B,2}(c) \lambda^4 \cdot 2\Delta \mathcal{E} + r_A(c) c^3 C_{B,3}(c) \lambda^2 \mathcal{H} \\ &\leq \left(C_A(c) \Phi_B(c) + r_A(c) c^3 C_{B,1}(c) + 2r_A(c) c^4 C_{B,2}(c) \right) \lambda^3 \mathcal{E} + r_A(c) c^3 C_{B,3}(c) \lambda^2 \mathcal{H}. \end{aligned} \quad (6.20)$$

Conclusion. Combining (6.13), (6.14), (6.15), (6.19), and (6.20), we arrive at the promised bound

$$\boxed{|R_{\geq 5}(\lambda)| \leq \Psi(c) \lambda^3 \mathcal{E} + \Xi(c) \lambda^2 \mathcal{H}(\lambda)}, \quad (6.21)$$

where

$$\begin{aligned} \Psi(c) &:= (11c^2 + 23c^3) + \left(M_B(c) C_A(c) + r_A(c) (1 + 2c) c^3 \right) \\ &\quad + \left((2 + 9c) \Phi_B(c) + M_A(c) (C_{B,1}(c) + 2c C_{B,2}(c)) \right) \\ &\quad + \left(C_A(c) \Phi_B(c) + r_A(c) c^3 (C_{B,1}(c) + 2c C_{B,2}(c)) \right) \end{aligned}$$

and

$$\Xi(c) := C_{B,3}(c) \left(M_A(c) + r_A(c) c^3 \right).$$

6.4 Bounding $\mathcal{H}(\lambda)$ by $\lambda \mathcal{E}$

Recall

$$\mathcal{H}(\lambda) = \sum_{uv \in E} |d_u - d_v| |R_u(F) - R_v(F)|, \quad \mathcal{E} = \sum_{uv \in E} (d_u - d_v)^2,$$

and

$$R_u(F) = \lambda^3 \sum_{x \in N(u)} \frac{d_x^2}{1 + d_x \lambda}.$$

Lemma 12 (A bound at the $\lambda\mathcal{E}$ scale for $\mathcal{H}(\lambda)$).

$$\mathcal{H}(\lambda) \leq \Gamma(c) \lambda \mathcal{E}, \quad \text{where } \Gamma(c) := (11 + 4c)c^2.$$

Proof. Define

$$h(t) := \frac{t^2}{1 + t\lambda}$$

for $t \geq 0$, so that $R_u(F) = \lambda^3 \sum_{x \in N(u)} h(d_x)$.

Step 1: A Taylor expansion of $R_u(F)$. We compute

$$h'(t) = \frac{t(2 + t\lambda)}{(1 + t\lambda)^2}, \quad h''(t) = \frac{2}{(1 + t\lambda)^3} \leq 2.$$

Fix $u \in V$. For each $x \in N(u)$, Taylor's theorem (with remainder) gives

$$h(d_x) = h(d_u) + h'(d_u)(d_x - d_u) + \frac{1}{2}h''(\xi_{ux})(d_x - d_u)^2$$

for some ξ_{ux} lying between d_u and d_x . Since $\frac{1}{2}h''(\xi_{ux}) \leq 1$, summing over $x \in N(u)$ yields

$$\sum_{x \in N(u)} h(d_x) = d_u h(d_u) + h'(d_u) \sum_{x \in N(u)} (d_x - d_u) + \varepsilon_u, \quad |\varepsilon_u| \leq \sum_{x \in N(u)} (d_x - d_u)^2.$$

Using $\sum_{x \in N(u)} (d_x - d_u) = -\sum_{x \in N(u)} (d_u - d_x) = -H_u$, we obtain

$$R_u(F) = \lambda^3 (d_u h(d_u) - h'(d_u) H_u) + \lambda^3 \varepsilon_u, \quad |\varepsilon_u| \leq V_u, \quad (6.22)$$

where

$$V_u := \sum_{x \in N(u)} (d_x - d_u)^2.$$

Step 2: A bound on $|R_u(F) - R_v(F)|$. Let

$$F(d) := d h(d) = \frac{d^3}{1 + d\lambda}, \quad G(d) := h'(d) = \frac{d(2 + d\lambda)}{(1 + d\lambda)^2}.$$

From (6.22),

$$|R_u(F) - R_v(F)| \leq \lambda^3 |F(d_u) - F(d_v)| + \lambda^3 |G(d_u) H_u - G(d_v) H_v| + \lambda^3 (|\varepsilon_u| + |\varepsilon_v|). \quad (6.23)$$

We bound each term.

(i) *The F term.* Differentiating F gives

$$F'(d) = \frac{d^2(3 + 2d\lambda)}{(1 + d\lambda)^2} \leq d^2(3 + 2d\lambda) \leq \Delta^2(3 + 2c).$$

Hence

$$\lambda^3 |F(d_u) - F(d_v)| \leq \lambda^3 \cdot \Delta^2(3 + 2c) \cdot |d_u - d_v| \leq (3 + 2c)c^2 \lambda |d_u - d_v|. \quad (6.24)$$

(ii) *The G term.* We use

$$|G(d_u) H_u - G(d_v) H_v| \leq |G(d_u) - G(d_v)| |H_u| + |G(d_v)| |H_u - H_v|.$$

First, since $G'(d) = h''(d) = 2/(1 + d\lambda)^3 \leq 2$, we have

$$|G(d_u) - G(d_v)| \leq 2|d_u - d_v|.$$

Also, $|H_u| \leq \Delta^2$, and

$$|G(d_v)| = \frac{d_v(2 + d_v\lambda)}{(1 + d_v\lambda)^2} \leq d_v(2 + d_v\lambda) \leq \Delta(2 + c).$$

Therefore

$$\begin{aligned} \lambda^3 |G(d_u) H_u - G(d_v) H_v| &\leq \lambda^3 \cdot 2|d_u - d_v| \cdot \Delta^2 + \lambda^3 \cdot \Delta(2 + c) |H_u - H_v| \\ &= 2c^2 \lambda |d_u - d_v| + (2 + c)c \lambda^2 |H_u - H_v|. \end{aligned} \quad (6.25)$$

(iii) *The Taylor-error term.* From (6.22), $|\varepsilon_u| \leq V_u$, so

$$\lambda^3(|\varepsilon_u| + |\varepsilon_v|) \leq \lambda^3(V_u + V_v). \quad (6.26)$$

Putting (6.24), (6.25), and (6.26) into (6.23),

$$|R_u(F) - R_v(F)| \leq (5 + 2c)c^2\lambda|d_u - d_v| + (2 + c)c\lambda^2|H_u - H_v| + \lambda^3(V_u + V_v). \quad (6.27)$$

Step 3: A bound on $\sum_{uv \in E} |d_u - d_v|(V_u + V_v)$. From

$$\sum_{uv \in E} (V_u + V_v) = \sum_{u \in V} d_u V_u \leq \Delta \sum_{u \in V} V_u$$

and

$$\sum_{u \in V} V_u = \sum_{u \in V} \sum_{x \in N(u)} (d_x - d_u)^2 = 2 \sum_{uv \in E} (d_u - d_v)^2 = 2\mathcal{E},$$

using $|d_u - d_v| \leq \Delta$, we get

$$\sum_{uv \in E} |d_u - d_v|(V_u + V_v) \leq \Delta \cdot \Delta \cdot 2\mathcal{E} = 2\Delta^2\mathcal{E}. \quad (6.28)$$

Combining (6.27), (6.28), and Lemma 8 yields

$$\begin{aligned} \mathcal{H}(\lambda) &= \sum_{uv \in E} |d_u - d_v| |R_u(F) - R_v(F)| \\ &\leq (5 + 2c)c^2\lambda \sum_{uv \in E} |d_u - d_v|^2 + (2 + c)c\lambda^2 \sum_{uv \in E} |d_u - d_v| |H_u - H_v| + \lambda^3 \sum_{uv \in E} |d_u - d_v|(V_u + V_v) \\ &\leq (5 + 2c)c^2\lambda\mathcal{E} + (2 + c)c\lambda^2 \cdot 2\Delta\mathcal{E} + \lambda^3 \cdot 2\Delta^2\mathcal{E} \\ &\leq (5 + 2c)c^2\lambda\mathcal{E} + 2(2 + c)c^2\lambda\mathcal{E} + 2c^2\lambda\mathcal{E} \\ &= (11 + 4c)c^2\lambda\mathcal{E}, \end{aligned}$$

which proves the lemma. □

7 Conclusion

(5.4) states

$$\mathcal{S}(\lambda) = 3\lambda^3\mathcal{E} + \lambda^4(\|Ld\|_2^2 - W) + R_{\geq 5}(\lambda).$$

By Lemma 7 and since $\Delta \geq 1$,

$$0 \leq W \leq 10\Delta\mathcal{E}.$$

(6.21) states

$$|R_{\geq 5}(\lambda)| \leq \Psi(c)\lambda^3\mathcal{E} + \Xi(c)\lambda^2\mathcal{H}(\lambda).$$

Also, Lemma 12 gives

$$\mathcal{H}(\lambda) \leq \Gamma(c)\lambda\mathcal{E}.$$

Combining,

$$\mathcal{S}(\lambda) \geq (3 - \Psi(c) - \Xi(c)\Gamma(c) - 10c)\lambda^3\mathcal{E}. \quad (7.1)$$

Recall that

$$\begin{aligned} \Psi(c) &= (11c^2 + 23c^3) + \left(M_B(c)C_A(c) + r_A(c)(1 + 2c)c^3 \right) \\ &\quad + \left((2 + 9c)\Phi_B(c) + M_A(c)(C_{B,1}(c) + 2cC_{B,2}(c)) \right) \\ &\quad + \left(C_A(c)\Phi_B(c) + r_A(c)c^3(C_{B,1}(c) + 2cC_{B,2}(c)) \right), \end{aligned}$$

$$\Xi(c) = C_{B,3}(c)\left(M_A(c) + (3c + 2c^2)^3 \right), \quad \Gamma(c) = (11 + 4c)c^2.$$

As $c \rightarrow 0$,

$$\Phi_B(c) = \frac{c^3}{1-c} + 3c^3 + 2c^4 \rightarrow 0, \quad (\text{Equation (3.6)})$$

$$r_A(c) \rightarrow 39, \quad (\text{Lemma 3})$$

$$C_A(c) = 1663c^2 \rightarrow 0 \quad (\text{Lemma 10})$$

$$C_{B,1}(c) = 1 + \frac{1+2c}{(1-c)^2} \rightarrow 2, \quad (\text{Lemma 11})$$

$$C_{B,2}(c) = 1 + \frac{1}{(1-c)^2} \rightarrow 2, \quad (\text{Lemma 11})$$

$$C_{B,3}(c) = \frac{1}{(1-c)^2} \rightarrow 1, \quad (\text{Lemma 11})$$

$$M_A(c) = 1 + 3c + 7c^2 \rightarrow 1, \quad (\text{Equation (6.12)})$$

$$M_B(c) = 1 + c + c^2 \rightarrow 1. \quad (\text{Equation (6.12)})$$

Therefore, as $c \rightarrow 0$, we have $\Psi(c) \rightarrow 2$, $\Xi(c) \rightarrow 1$, and $\Gamma(c) \rightarrow 0$. Hence, by (7.1), there exists $c \in (0, 1)$ such that $\mathcal{S}(\lambda) \geq 0$, which can in fact be chosen as $c = 0.0179362$. This concludes the proof.

Acknowledgments

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References

- [DSST26] Ewan Davies, Juspreet Singh Sandhu, Jaehyeon Seo, and Brian Tan. *Degree-sequence bounds for independent sets via multivariate local occupancy*. 2026. arXiv: [2605.05149](#) [math.CO] (cit. on p. 2).
- [DST25] Ewan Davies, Juspreet Singh Sandhu, and Brian Tan. *On expectations and variances in the hard-core model on bounded degree graphs*. 2025. arXiv: [2505.13396](#) [math.CO] (cit. on pp. 1, 2).